FERMILAB-SLIDES-24-0174-T

From Amplitudes to Experiments

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Lecture notes for TASI Summer School 2024

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Goals of this lecture

Bird's eye overview

- Event generators have been a topic of intense research for decades, as they connect theory and experiment and need to make concessions to both
- There is renewed interest in understanding their interplay with analytic resummation, and in finding new and better algorithms that allow an extension to higher formal accuracy
- To some extent, the definition of accuracy itself is still being worked on

What to expect

- The background that allows you to understand what is being discussed in past and present event generator literature, and why
- The tutorial as a chance for in-depth discussion of the basic concepts presented in the lecture (matrix elements, parton showers & resummation)

What not to expect

- All the latest and greatest plots, as well as a survey of all possible algorithms. This could fill the entire time of the school.

Suggested reading

1. R. K. Ellis, W. J. Stirling, B. R. Webber
   QCD and Collider Physics
   Cambridge University Press, 2003
2. R. D. Field
   Applications of Perturbative QCD
   Addison-Wesley, 1995
3. M. E. Peskin, D. V. Schroeder
   An Introduction to Quantum Field Theory
   Westview Press, 1995
4. L. Dixon, F. Petriello (Editors)
   Journeys Through the Precision Frontier
   Proceedings of TASI 2014, World Scientific, 2015
5. T. Sjöstrand, S. Mrenna, P. Z. Skands
   PYTHIA 6.4 Physics and Manual
   JHEP 05 (2006) 026

Additional references provided on the slides
Only if material not covered in these books

Hands on tutorials

- Resource for learning more about parton showers: Live "Hackathon"

  `git clone https://gitlab.com/shoeche/tutorials.git`

MC tutorials at the 2024 TASI school

Getting started

This tutorial uses a container. Please install Docker on your personal computer prior to the tutorial. If you have questions on installing Docker or on testing the Docker e
cannot assist everyone with setting up Docl

Download

The Docker container can be pulled directly

```
docker pull cteqschool/tutorial
```

Running the container

We recommend to run the container for the
to simplify the docker call you can define a

```
alias docker-run='docker run -i
```

You may consider adding the --rm flag to a
you may have made to the container will be

Running the tutorials

Instructions for the tutorials are found in thi
matrix-element, resummation and parton-sl

Introduction to Parton Showers and Matching

Tutorial for summer schools

1 Introduction

In this tutorial we will discuss the construction of a parton shower, the implementation of on-the-fly uncertainty estimates, and of matrix-element corrections, and matching at next-to-leading order. At the end, you will be able to run your own parton shower for $e^+e^- \to$hadrons at LEP energies and compare its predictions to results from the event generator Sherpa (using a simplified setup). You will also have constructed your first MC@NLO and POWHEG generator.

2 Getting started

You can use any of the docker containers for the school to run this tutorial. Should you have problems with disk space, consider running `docker containers prune` and `docker system prune` first. To launch the docker container, use the following command

```
docker run -it -u $(id -u $USER) --rm -v $HOME:$HOME -w $PWD <container name>
```

You can also use your own PC (In this case you should have PyPy and Rivet installed). Download the tutorial and change to the relevant directory by running

```
git clone https://gitlab.com/shoeche/tutorials.git && cd tutorials/ps/
```

For simplicity, this tutorial uses PyPy, a just-in-time compiled variant of Python. If you are unfamiliar

Outline of lectures

- Monte Carlo methods
- Hard matrix elements
- Many-body phase space
- Radiative corrections
- Technical interlude
- Semi-classical picture
- Color coherence
- Higher-order effects
- Connection to resummation
- Forward vs. backward evolution
- Matching to higher orders
- Multi-jet merging

Monte Carlo methods

Stating the problem

- Want to compute expectation values of observables

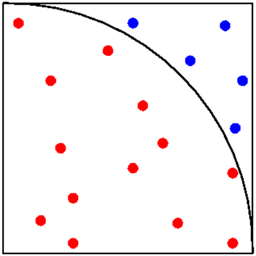
  $$\langle O \rangle \;=\; \sum_n \int \mathrm{d}\Phi_n \, P(\Phi_n) \, O(\Phi_n)$$

  Φ_n - Point in n-particle phase-space
  $P(\Phi_n)$ - Probability to produce Φ_n
  $O(\Phi_n)$ - Value of observable at Φ_n

- Problem $\#1$: Computing $P(\Phi_n)$
- Problem $\#2$: Performing the integral

- At lower orders in perturbation theory, problem $\#2$ is harder
  This is where event generators come into play

The hit-or-miss method

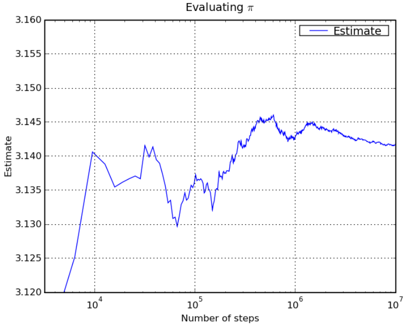


$$\frac{\text{Hits}}{\text{Misses} + \text{Hits}} \longrightarrow \frac{\pi}{4}$$

Throw random points (x,y),

with x, y in [0,1]

For hits: $(x^2 + y^2) < r^2 = 1$



Importance sampling

- In many cases we can approximate the integral of $f(x)$ with some known function $g(x)$ such that primitive $G(x)$ is known
- This amounts to a variable transformation

$$I = \int_a^b \mathrm{d}x\, g(x)\, \frac{f(x)}{g(x)} = \int_{G(a)}^{G(b)} \mathrm{d}G(x)\, w(x) \quad \text{where} \quad w(x) = \frac{f(x)}{g(x)}$$

- Integral and error estimate are

$$I = [G(b) - G(a)]\langle w \rangle \qquad\qquad \sigma = [G(b) - G(a)]\sqrt{\frac{\langle w^2 \rangle - \langle w \rangle^2}{N-1}}$$

  N - Number of MC events (points)
- MC error scales as $1/\sqrt{N}$ independent of number of dimensions!
- Note that I is independent of $g(x)$, but σ is not $\rightarrow$ suitable choice of $g(x)$ can be used to minimize error

Selection from a known distribution

- Random number generators produce uniform pseudo-random numbers in $[0, 1]$
- Assume we want points following the distribution $g(x)$ with known primitive $G(x)$ instead
- Probability of producing point in $[x, x + \mathrm{d}x]$ is $g(x)\,\mathrm{d}x$
- Can generate x according to

  $$\int_a^x \mathrm{d}x'\, g(x') = R \int_a^b \mathrm{d}x'\, g(x')$$

  where R is a uniform random number in $[0, 1]$

  $$x = G^{-1}\Big[G(a) + R\big(G(b) - G(a)\big)\Big]$$

Hard matrix elements

Helicity

[Dixon] hep-ph/9601359, [Dittmaier] hep-ph/9805445

- Weyl-van-der-Waerden spinors for helicity states $+/-$

$$\chi_+(p) = \begin{pmatrix} \sqrt{p^+} \\ \sqrt{p^-}e^{i\phi_p} \end{pmatrix} \qquad \chi_-(p) = \begin{pmatrix} \sqrt{p^-}e^{i\phi_p} \\ -\sqrt{p^+} \end{pmatrix} \qquad \begin{array}{rcl} p^\pm &=& p^0 \pm p^3 \\ p_\perp &=& p^1 + ip^2 \end{array}$$

  Basic building blocks for all amplitudes
  $+, -, \perp$ directions define "spinor gauge"
- Massive Dirac spinors in terms of WvdW spinors

$$u_+(p,m) = \frac{1}{\sqrt{2\,\bar{p}}} \begin{pmatrix} \sqrt{p_0 - \bar{p}}\,\chi_+(\hat{p}) \\ \sqrt{p_0 + \bar{p}}\,\chi_+(\hat{p}) \end{pmatrix} \qquad \bar{p} = \mathrm{sgn}(p_0)\,|\vec{p}|$$

$$u_-(p,m) = \frac{1}{\sqrt{2\,\bar{p}}} \begin{pmatrix} \sqrt{p_0 + \bar{p}}\,\chi_-(\hat{p}) \\ \sqrt{p_0 - \bar{p}}\,\chi_-(\hat{p}) \end{pmatrix} \qquad \hat{p} = (\bar{p}, \vec{p})$$

- γ^5 conveniently defined in Weyl representation

$$\gamma^5 \;=\; i\gamma^0\gamma^1\gamma^2\gamma^3 \;=\; \begin{pmatrix} -\sigma^0 & 0 \\ 0 & \sigma^0 \end{pmatrix}$$

  Projection operator $P_{R,L} = P_\pm = (1 \pm \gamma^5)/2$ identifies
  lower/upper component of Dirac spinors as right-/left-handed

Helicity

- Massless polarizations constructed from $u_\pm(p)$ and $u_\pm(k)$ with external light-like gauge vector k
$$\varepsilon^\mu_\pm(p,k) = \pm \frac{\bar{u}_\mp(k)\gamma^\mu u_\mp(p)}{\sqrt{2}\,\bar{u}_\mp(k)u_\pm(p)}\,.$$
Defines light-like axial gauge
- For massive particles decompose momentum p using k
$$b = p - \kappa k \qquad \kappa = \frac{p^2}{2pk} \qquad \Rightarrow \qquad b^2 = 0$$
Transverse polarizations as in massless case ($p \to b$) plus longitudinal
$$\varepsilon^\mu_0(p,k) = \frac{1}{m}\left(\bar{u}_-(b)\gamma^\mu u_-(b) - \kappa\,\bar{u}_-(k)\gamma^\mu u_-(k)\right)$$
- Vertices & propagators have simpler structure
- Building blocks for Standard model complete!

Color

[Maltoni,Stelzer,Willenbrock] hep-ph/0209271, [Duhr,SH,Maltoni] hep-ph/0607057

- QCD amplitudes can be stripped of color factors
- Fundamental representation for n-gluons

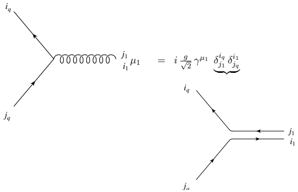
$$\mathcal{A}_n(p_1,\ldots,p_n) = \sum_{\vec{\sigma}\in P(2,\ldots,n)} \mathrm{Tr}(\lambda^{a_1}\lambda^{a_{\sigma_2}}\ldots\lambda^{a_{\sigma_n}})\, A(p_1,p_{\sigma_2},\ldots,p_{\sigma_n})$$

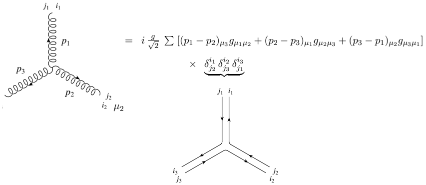
- Adjoint representation for n-gluons

$$\mathcal{A}_n(p_1,\ldots,p_n) = \sum_{\vec{\sigma}\in P(2,\ldots,n-1)} \left[F^{a_{\sigma_2}}\ldots F^{a_{\sigma_{n-1}}}\right]_{a_n}^{a_1} A(p_1,p_{\sigma_2},\ldots,p_{\sigma_{n-1}},p_n)$$

- Color-flow representation for n-gluons

$$\mathcal{A}_n(p_1,\ldots,p_n) = \sum_{\vec{\sigma}\in P(2,\ldots,n)} \delta^{i_1}_{j_{\sigma_2}}\delta^{i_{\sigma_2}}_{j_{\sigma_3}}\ldots\delta^{i_{\sigma_n}}_{j_1}\; A(p_1,p_{\sigma_2},\ldots,p_{\sigma_n})$$





Color

- We can sample colors just like we sample momenta
- Assign one in (r, g, b) / $(\bar{r}, \bar{g}, \bar{b})$ to each external (anti-)quark & gluon
- Average number of partial amplitudes is then smallest in color-flow basis

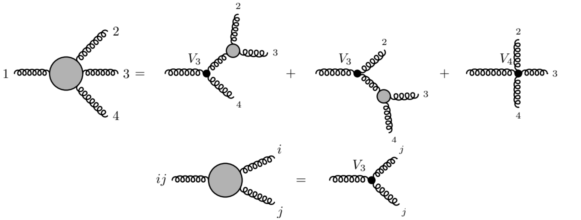
n	Average # of partials Gell-Mann	Color-flow	Adjoint
4	4.83	1.28	1.15
5	15.2	1.83	1.52
6	56.5	3.21	2.55
7	251	6.80	5.53
8	1280	17.0	15.8
9	7440	48.7	56.4
10	47800	158	243

n	Time [s/10^4pt] CO	CD
4	1.20	1.04
5	3.78	2.69
6	14.2	7.19
7	58.5	23.7
8	276	82.1
9	1450	270
10	7960	864

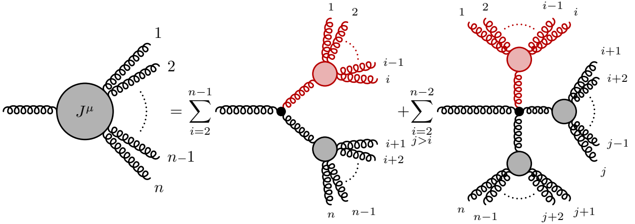
- Computational effort reduced further by not stripping amplitudes of color factors
- Evaluate dynamically at each vertex $\rightarrow$ straightforward computer algorithm
- Color dressing (CD) vs. color ordering (CO)

Amplitude construction

Example: Diagrams for
$g(1)g(2) \to g(3)g(4)$



[Berends,Giele] NPB306(1988)759



Example: Currents for
$g(1)g(2) \to g(3)g(4)$

Step 1	$J_1 = \varepsilon(1)$	$J_2 = \varepsilon(2)$	$J_3 = \varepsilon(3)$	$J_4 = \varepsilon(4)$
Step 2	J_{12}	J_{13}	J_{23}	
Step 3	J_{123}			
Step 4	$A(1,2,3,4) = J_4^*\, J_{123}$			

Many-body phase space

Phase space

[James] CERN-68-15, [Byckling,Kajantie] NPB9(1969)568

- Need to evaluate in a process-independent way

$$\mathrm{d}\Phi_n(p_a, p_b; p_1, .., p_n) \;=\; \left[\prod_{i=1}^{n} \frac{\mathrm{d}^3 p_i}{(2\pi)^3\, 2E_i}\right] \delta^4\left(p_a + p_b - \sum_{i=1}^{n} p_n\right)$$

- Use factorization properties of phase-space integral

$$\begin{aligned}\mathrm{d}\Phi_n(p_a, p_b; p_1, .., p_n) \;&= \mathrm{d}\Phi_{n-m+1}(p_a, p_b; p_{1m}, p_{m+1}, .., p_n)\\ &\times \frac{\mathrm{d}s_{1m}}{2\pi}\, \mathrm{d}\Phi_m(p_{1m}; p_1, .., p_m)\end{aligned}$$

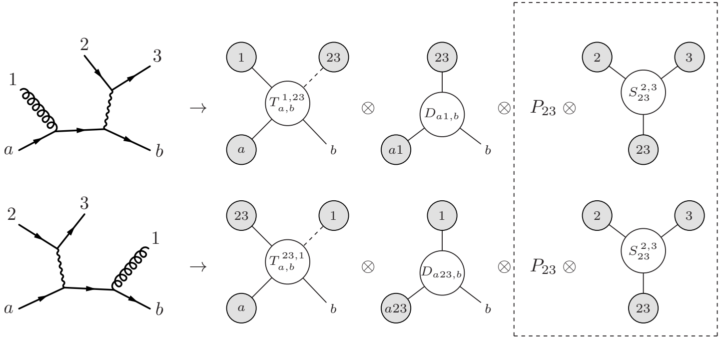
- Apply repeatedly until only 2-particle phase space elements remain
- In the frame of a time-like momentum, P, this gives:

$$\mathrm{d}\Phi_2(p_{12}; p_1, p_2) = \frac{1}{16\pi^2} \frac{\sqrt{(p_1 P)^2 - p_1^2 P^2}^{\,3}}{((p_1 P)(p_1 p_2) - p_1^2 (p_2 P))P^2}\, \mathrm{d}\cos\theta_1^{(P)} \mathrm{d}\phi_1^{(P)}$$

- Typically evaluated in center-of-mass frame of combined momentum, $p_1 + p_2$

$$\mathrm{d}\Phi_2(p_{12}; p_1, p_2) = \frac{1}{16\pi^2} \frac{\sqrt{(p_1 p_2)^2 - p_1^2 p_2^2}}{(p_1 + p_2)^2}\, \mathrm{d}\cos\theta_1^{\{1,2\}}\, \mathrm{d}\phi_1^{\{1,2\}}$$

Phase space – Diagram based



- Construct one integrator per diagram and combine into multi-channel
- Intuitive notion of pole structure, multi-channel determines balance
- Factorial growth with number of diagrams can be tamed by recursion

Phase space – T-channel dominated

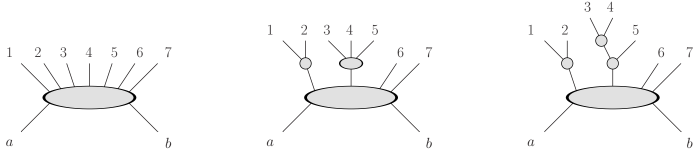
- At hadron colliders, convenient form of single-particle given by

$$\frac{\mathrm{d}^3\vec{p}_i}{(2\pi)^3\,2E_i} = \frac{1}{16\pi^2}\,\mathrm{d}p_{i,\perp}^2\,\mathrm{d}y_i\,\frac{\mathrm{d}\phi_\mathrm{i}}{2\pi}$$

- Scalable t-channel parametrization of full phase-space integral

$$\mathrm{d}x_a\mathrm{d}x_b\,\mathrm{d}\Phi_n(a,b;1,\ldots,n) = \frac{2\pi}{s}\left[\prod_{i=1}^{n-1}\frac{1}{16\pi^2}\,\mathrm{d}p_{i,\perp}^2\,\mathrm{d}y_i\,\frac{\mathrm{d}\phi_\mathrm{i}}{2\pi}\right]\mathrm{d}y_n$$

- Can be combined with s-channel integrators to improve convergence



- Ideally suited as a starting point for ML-based improvements

Improving efficiency with Neural Networks

Surrogate model techniques

- Hit or miss w/ NN estimate
  arXiv:2109.11964
  - An order of magnitude faster
  - Insufficient training yields large uncertainties, but no bias
  - Needs existing sample to train
- Generate events with GANs
  arXiv:1707.00028, arXiv:1901.00875, arXiv:1901.05282, arXiv:1903.02433, arXiv:1907.03764, arXiv:1912.08824, arXiv:1909.01359, arXiv:1909.04451, ...
  - Orders of magnitude faster
  - Needs existing sample to train
  - Bias if not trained right

Variabe transformation technique

- Learn integrand to improve importance sampling
  arXiv:1707.00028, arXiv:1810.11509, arXiv:2001.05478 arXiv:2001.05486
- Insufficient training yields large uncertainties, but no bias
- Events generated from scratch no pre-existing sample required
- Resulting events still need to be unweighted

Event generation with Normalizing Flows

- Straightforward MC integral estimator

$$I = \int_\Omega f(x)\,\mathrm{d}x = \frac{\Omega}{N}\sum_{i=1}^{N} f(x_i) = \Omega\,\langle f\rangle_x\ , \qquad \sigma_I = \Omega\sqrt{\frac{\langle f^2\rangle_x - \langle f\rangle_x^2}{N-1}}$$

- After variable transformation $\mathrm{d}x \to \mathrm{d}x\, g(x) = \mathrm{d}G(x)$

$$I = \int_\Omega \frac{f(x)}{g(x)}\,\mathrm{d}G(x) = \Omega\,\langle f/g\rangle_G\ , \qquad \sigma_I = \Omega\sqrt{\frac{\langle (f/g)^2\rangle_G - \langle f/g\rangle_G^2}{N-1}}$$

- Multi-dimensional integrals: $\mathrm{d}\vec{x} \to \mathrm{d}\vec{x}'\,|\mathrm{d}\vec{x}(\vec{x}')/\mathrm{d}\vec{x}'|$
  $\to$ Jacobian changes from $1/g(x)$ to $|\mathrm{d}\vec{x}'/\mathrm{d}\vec{x}|^{-1}$
- For bijective map g of random variable $\vec{x}$ drawn from base distribution q_0, the variable $\vec{x}' = g(\vec{x})$ follows distribution inferred by chain rule:

$$q_1(\vec{x}') = q_0(g^{-1}(\vec{x}'))\,|\left|\frac{\partial g^{-1}(\vec{x}')}{\partial \vec{x}'}\right|| = q_0(\vec{x})\,|\left|\frac{\partial g(\vec{x})}{\partial \vec{x}}\right|^{-1}|$$

- In CS literature, this transformation is called a “Normalizing Flow”

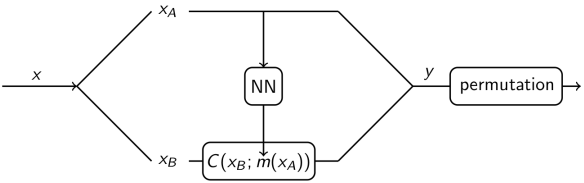
Event generation with Normalizing Flows

- If we use a Neural Network to learn g, we need to compute its gradient during inference. This is veeery slow. No, really! It's even slower!
- "Coupling layers" are special bijectors to avoid these gradients:
  - Input variables $\vec{x} = \{x_1, .., x_D\}$ partitioned into two subsets, $\vec{x}_A$ and $\vec{x}_B$

    $$x'_A = x_A, \qquad x'_B = C(x_B; m(\vec{x}_A))$$

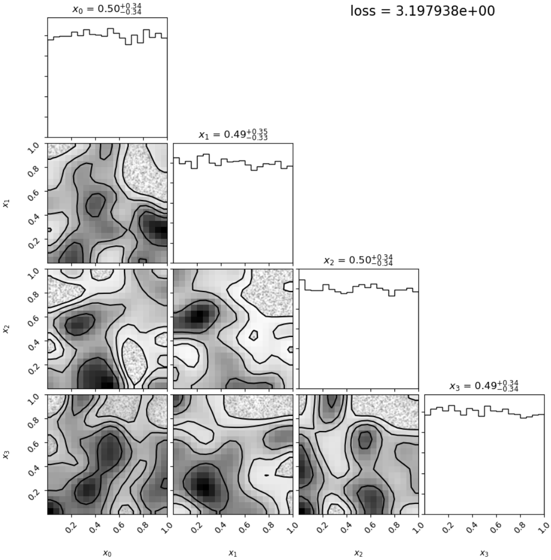
  - m is output of a Neural Network taking x_A as inputs and returning parameters of "Coupling Transform" C that will be applied to x_B
  - Inverse map is simple, leading to simple Jacobian (no $\partial m / \partial \vec{x}_A$!)

$$\left|\frac{\partial g(\vec{x})}{\partial \vec{x}}\right|^{-1} = \left|\begin{pmatrix} \vec{1} & 0 \\ \frac{\partial C}{\partial m}\frac{\partial m}{\partial \vec{x}_A} & \frac{\partial C}{\partial \vec{x}_B} \end{pmatrix}\right|^{-1} = \left|\frac{\partial C(\vec{x}_B; m(\vec{x}_A))}{\partial \vec{x}_B}\right|^{-1} .$$



Toy example - 4-dimensional camel function

[Gao,Isaacson,Krause] arXiv:2001.05486



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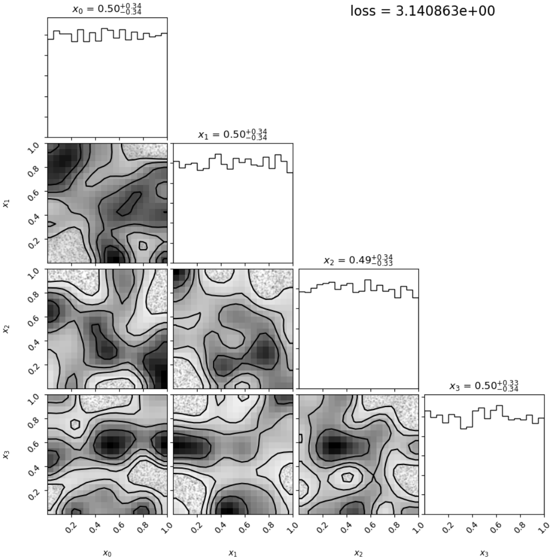

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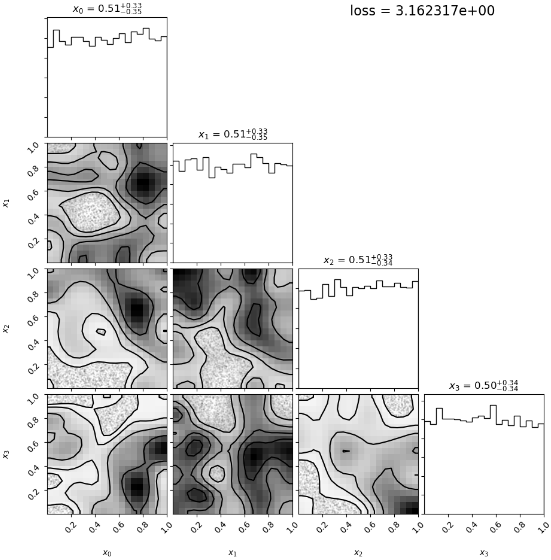

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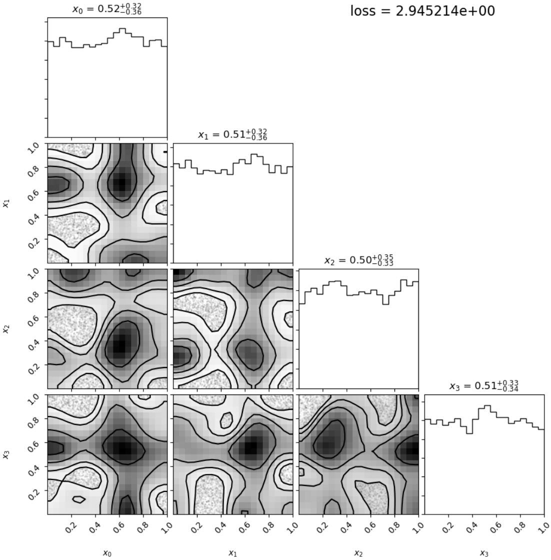

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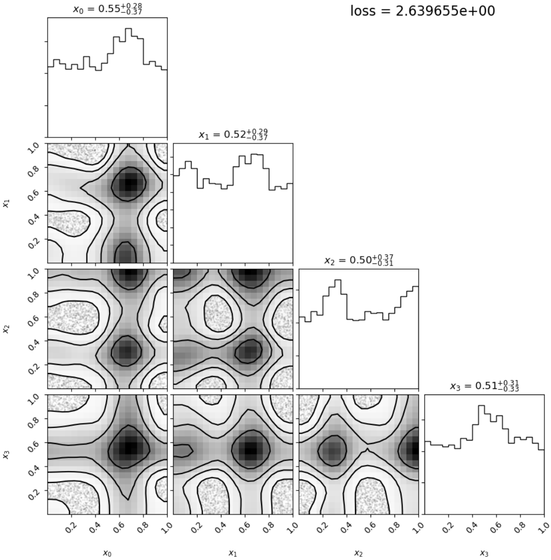

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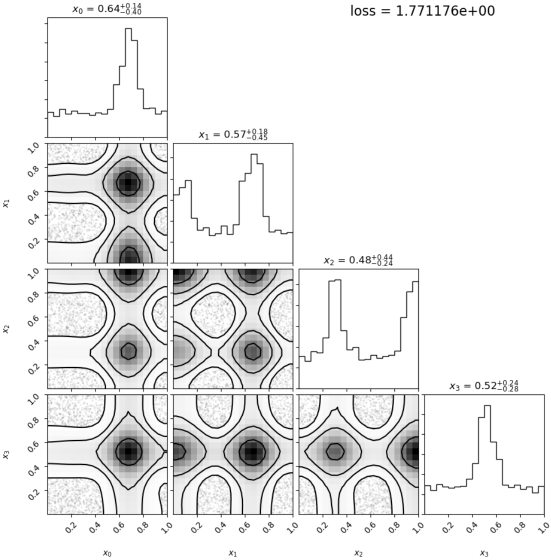

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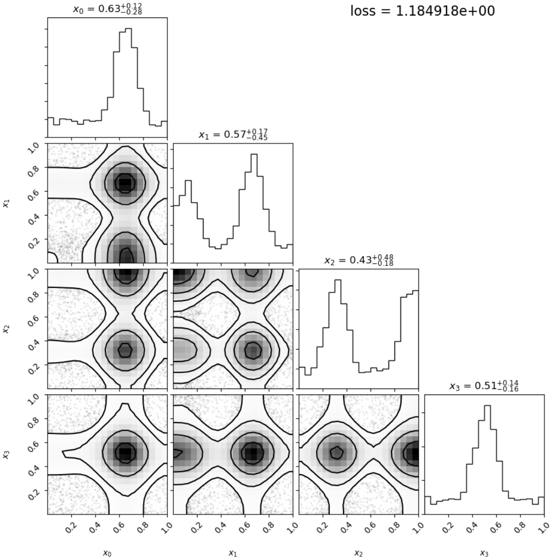

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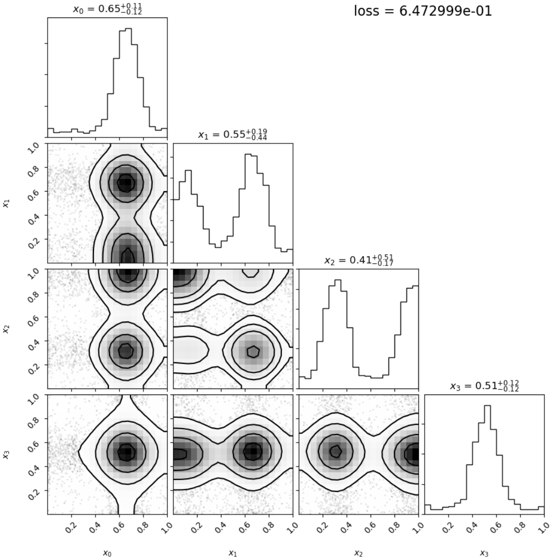

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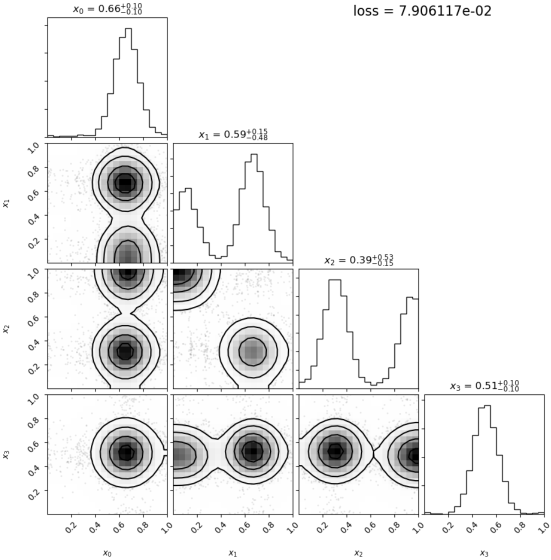

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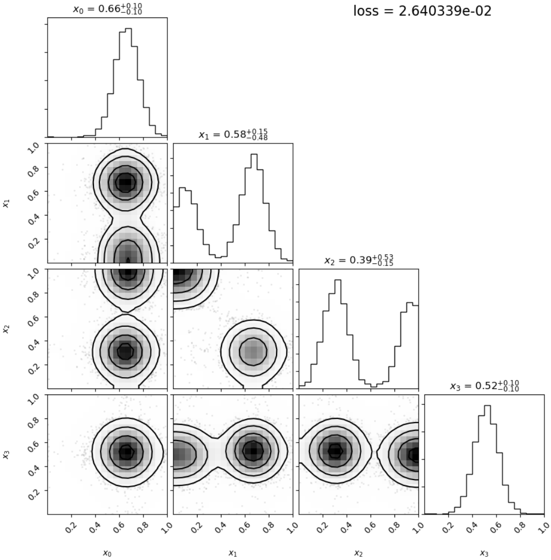
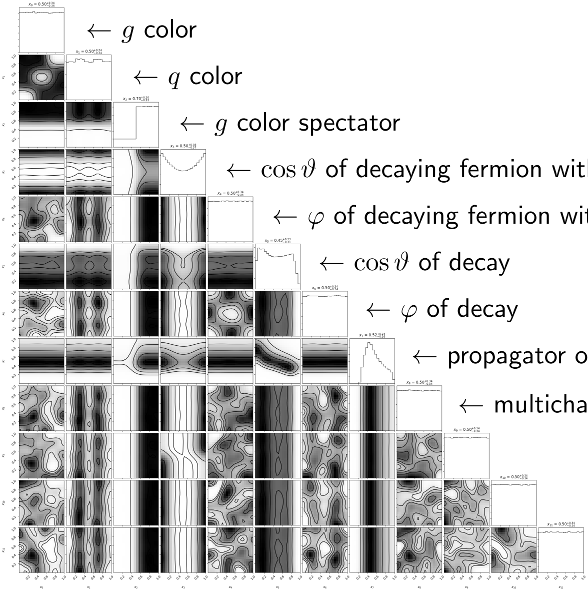


Real-life example: $e^+e^- \to qqg$

[Gao,Isaacson,Krause,Schulz,SH] arXiv:2001.10028

Target distribution with learning color



$\leftarrow g$ color

$\leftarrow q$ color

$\leftarrow g$ color spectator

$\leftarrow \cos\vartheta$ of decaying fermion with beam

$\leftarrow \varphi$ of decaying fermion with beam

$\leftarrow \cos\vartheta$ of decay

$\leftarrow \varphi$ of decay

$\leftarrow$ propagator of decaying fermion

$\leftarrow$ multichannel ...

Real-life example: $pp \to V + jets$

[Gao,Isaacson,Krause,Schulz,SH] arXiv:2001.10028

- Check results in most performance-critical applications
- To make unweighting efficiency independent of weight outliers, define max as median of maxima in bootstrap approach [Campbell,Neumann] arXiv:1909.09117

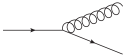
unweighting efficiency		LO QCD					NLO QCD (RS)	
$\langle w \rangle / w_{\max}$		n =0	n =1	n =2	n =3	n =4	n =0	n =1
$W^+ + n$ jets	Sherpa	$2.5 \cdot 10^{-1}$	$3.4 \cdot 10^{-2}$	$6.7 \cdot 10^{-3}$	$1.7 \cdot 10^{-3}$	$6.6 \cdot 10^{-4}$	$6.5 \cdot 10^{-2}$	$2.9 \cdot 10^{-3}$
	NN+NF	$5.8 \cdot 10^{-1}$	$1.2 \cdot 10^{-1}$	$8.8 \cdot 10^{-3}$	$1.6 \cdot 10^{-3}$	$8.9 \cdot 10^{-4}$	$1.7 \cdot 10^{-1}$	$4.0 \cdot 10^{-3}$
	Gain	2.3	3.6	1.3	0.99	1.4	2.7	1.4
$W^- + n$ jets	Sherpa	$2.4 \cdot 10^{-1}$	$3.9 \cdot 10^{-2}$	$8.4 \cdot 10^{-3}$	$1.7 \cdot 10^{-3}$	$8.8 \cdot 10^{-4}$	$6.0 \cdot 10^{-2}$	$3.3 \cdot 10^{-3}$
	NN+NF	$6.2 \cdot 10^{-1}$	$1.3 \cdot 10^{-1}$	$1.2 \cdot 10^{-2}$	$2.3 \cdot 10^{-3}$	$9.8 \cdot 10^{-4}$	$1.6 \cdot 10^{-1}$	$3.8 \cdot 10^{-3}$
	Gain	2.6	3.2	1.5	1.4	1.17	2.8	1.2
$Z + n$ jets	Sherpa	$4.3 \cdot 10^{-1}$	$4.3 \cdot 10^{-2}$	$1.3 \cdot 10^{-2}$	$2.7 \cdot 10^{-3}$	$1.1 \cdot 10^{-3}$	$1.1 \cdot 10^{-1}$	$4.9 \cdot 10^{-3}$
	NN+NF	$5.1 \cdot 10^{-1}$	$1.1 \cdot 10^{-1}$	$1.3 \cdot 10^{-2}$	$2.6 \cdot 10^{-3}$		$1.8 \cdot 10^{-3}$	$4.9 \cdot 10^{-3}$
	Gain	1.2	2.6	1.1	0.97		1.7	1.0

Radiative corrections

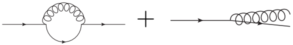
The heuristic view

Radiative corrections as a branching process

- Make two well motivated assumptions
  - Parton branching can occur in two ways



\- observed



\- unobserved

  - Evolution conserves probability
- The consequence is Poisson statistics
  - Let the decay probability be λ
  - Assume indistinguishable particles $\rightarrow$ naive probability for n emissions

$$P_{\text{naive}}(n,\lambda) = \frac{\lambda^n}{n!}$$

  - Probability conservation (i.e. unitarity) implies a no-emission probability

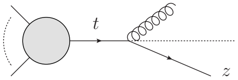
$$P(n,\lambda) = \frac{\lambda^n}{n!}\exp\{-\lambda\} \qquad \longrightarrow \qquad \sum_{n=0}^{\infty} P(n,\lambda) = 1$$

- In the context of parton showers $\Delta = \exp\{-\lambda\}$ is called a Sudakov factor

Radiative corrections as a branching process

- Decay probability for parton state in collinear limit

$$\lambda \to \frac{1}{\sigma_n} \int_t^{Q^2} \mathrm{d}\bar{t}\, \frac{\mathrm{d}\sigma_{n+1}}{\mathrm{d}\bar{t}} \approx \sum_{\text{jets}} \int_t^{Q^2} \frac{\mathrm{d}\bar{t}}{\bar{t}} \int \mathrm{d}z\, \frac{\alpha_s}{2\pi} P(z)$$



Parameter t identified with evolution "time"

- Splitting function $P(z)$ spin & color dependent

$$P_{qq}(z) = C_F \left[\frac{2z}{1-z} + (1-z) \right] \qquad P_{gq}(z) = T_R \left[z^2 + (1-z)^2 \right]$$

$$P_{gg}(z) = C_A \left[\frac{2z}{1-z} + z(1-z) \right] + (z \leftrightarrow 1-z)$$

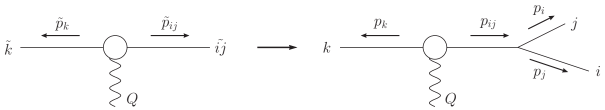
Exercise: Why does the $2z/(1-z)$ term appear both in P_{qq} and P_{gg}?

- When adding partons
  - On-shell conditions must be maintained
  - Overall four-momentum must be conserved
  - Color must be conserved
- Later in this lecture we will derive part of these splitting functions and analyze their properties

How to deal with the phase space

Example momentum mapping

Final state momentum mapping



- Generate off-shell momentum by rescaling

$$p_{ij}^\mu = \tilde{p}_{ij}^\mu + \frac{p_{ij}^2}{2\tilde{p}_{ij}\tilde{p}_k}\,\tilde{p}_k^\mu\,, \qquad p_k^\mu = \left(1 - \frac{p_{ij}^2}{2\tilde{p}_{ij}\tilde{p}_k}\right)\tilde{p}_k^\mu$$

- Then branch into two on-shell momenta

$$p_i^\mu = \tilde{z}\,\tilde{p}_{ij}^\mu + (1-\tilde{z})\,\frac{p_{ij}^2}{2\tilde{p}_{ij}\tilde{p}_k}\tilde{p}_k^\mu + k_\perp^\mu$$

$$p_j^\mu = (1-\tilde{z})\,\tilde{p}_{ij}^\mu + \tilde{z}\,\frac{p_{ij}^2}{2\tilde{p}_{ij}\tilde{p}_k}\tilde{p}_k^\mu - k_\perp^\mu$$

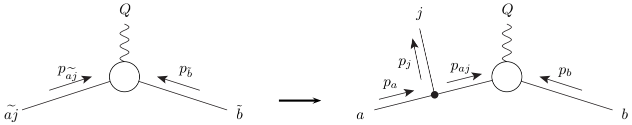
Exercise: Is this momentum mapping collinear safe?

- On-shell conditions require that

$$\vec{k}_T^2 = p_{ij}^2\,\tilde{z}(1-\tilde{z}) \qquad \leftrightarrow \qquad \tilde{z}_\pm = \frac{1}{2}\left(1 \pm \sqrt{1 - 4\vec{k}_T^2/p_{ij}^2}\right)$$

$\rightarrow$ for any finite $\vec{k}_T$ we have $0 < \tilde{z} < 1$

Initial state momentum mapping



- Rescale beam momentum to obtain new partonic cms energy

$$p_a^\mu = \frac{2\,p_a p_b}{2\,\tilde{p}_{aj}\tilde{p}_b}\,\tilde{p}_{aj}^\mu$$

- Compute final-state momentum and internal momentum

$$p_{aj}^\mu = \tilde{z}\,p_a^\mu + \frac{p_{aj}^2}{2p_b p_a}\,p_b^\mu + k_\perp^\mu$$

$$p_j^\mu = (1-\tilde{z})\,p_a^\mu - \frac{p_{aj}^2}{2p_b p_a}\,p_b^\mu - k_\perp^\mu$$

- Recoil taken by complete final state via Lorentz transformation

$$p_i^\mu = p_{\tilde{\imath}}^\mu - \frac{2\,p_{\tilde{\imath}}(K+\tilde{K})}{(K+\tilde{K})^2}\,(K+\tilde{K})^\mu + \frac{2\,p_{\tilde{\imath}}\tilde{K}}{\tilde{K}^2}\,K^\mu\;,$$

where $K^\mu = p_a^\mu - p_j^\mu + p_b^\mu$ and $\tilde{K}^\mu = p_{\widetilde{aj}}^\mu + p_b^\mu$

How to color a jet

The improved large-N_c approximation

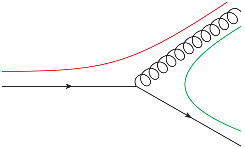
Color flow

- Write gluon propagator using completeness relations

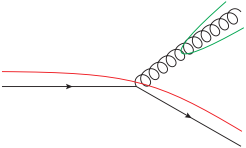
$$\underbrace{\delta^{ab}}_{\text{standard}} = 2\,\mathrm{Tr}(T^a T^b) = 2\,T^a_{ij} T^b_{ji} = T^a_{ij}\,\underbrace{2\,\delta_{ik}\delta_{jl}}_{\text{color flow}}\,T^b_{lk}$$

- Quark-gluon vertex

$$T^a_{ij} T^a_{kl} = \frac{1}{2}\left(\delta_{il}\delta_{jk} - \frac{1}{N_c}\delta_{ij}\delta_{kl}\right)$$

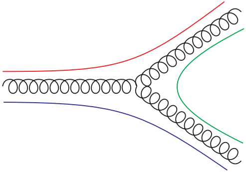


$$-\frac{1}{N_c}$$



- Gluon-gluon vertex

$$f^{abc} T^a_{ij} T^b_{kl} T^c_{mn} = \delta_{il}\delta_{kn}\delta_{mj} - \delta_{in}\delta_{ml}\delta_{kj}$$



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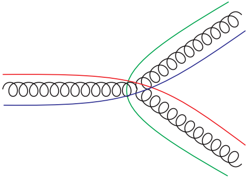
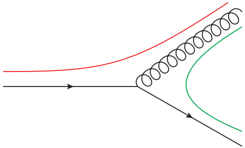


Exercise: Can you explain why there is no $1/N_c$ term here?

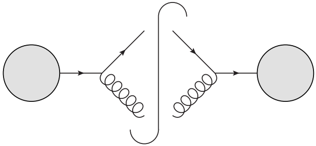
Color flow

- Typically, parton showers also make the leading-color approximation

$T^a_{ij} T^a_{kl} \to \frac{1}{2}\delta_{il}\delta_{jk} \qquad \leftrightarrow$



- If used naively, this would overestimate the color charge of the quark: Consider process $q \to qg$ attached to some larger diagram



$\propto \qquad T^a_{ij} T^a_{jk} = C_F \delta_{ik}$

but now we have $\frac{1}{2}\,\delta_{il}\delta_{jm}\delta_{mj}\delta_{lk} = \frac{C_A}{2}\,\delta_{ik}$

- Color assignments in parton shower made at leading color but color charge of quarks actually kept at C_F
  Exercise: How should colors be assigned when a gluon splits into two gluons?

How to implement the algorithm

Monte-Carlo methods for jet evolution

Monte-Carlo methods: Poisson distributions

- Assume decay process described by $g(t)$
- Decay can happen only if it has not happened already
  Must account for survival probability $\leftrightarrow$ Poisson distribution

$$\mathcal{G}(t) \;=\; g(t)\Delta(t,t_0) \qquad \text{where} \qquad \Delta(t,t_0) \;=\; \exp\left\{-\int_t^{t_0} \mathrm{d}t'\, g(t')\right\}$$

- If $G(t)$ is known, then we also know the integral of $\mathcal{G}(t)$

$$\int_t^{t_0} \mathrm{d}t' \mathcal{G}(t') \;=\; \int_t^b \mathrm{d}t' \;\frac{\mathrm{d}\Delta(t',t_0)}{\mathrm{d}t'} \;=\; 1-\Delta(t,t_0)$$

- Can generate events by requiring $1-\Delta(t,t_0) = 1-R$

$$t = G^{-1}\Big[G(t_0) + \log R\Big]$$

You will use this formula in the tutorial

Monte-Carlo methods: Poisson distributions

- Importance sampling for Poisson distributions
  - Generate event according to $\mathcal{G}(t)$
  - Accept with $w(t) = f(t)/g(t)$
  - If rejected, continue starting from t
- Probability for immediate acceptance

$$\frac{f(t)}{g(t)}\, g(t) \exp\left\{-\int_t^{t_0} \mathrm{d}t'\, g(t')\right\}$$

- Probability for acceptance after one rejection

$$\frac{f(t)}{g(t)}\, g(t) \int_t^{t_0} \mathrm{d}t_1 \exp\left\{-\int_t^{t_1} \mathrm{d}t'\, g(t')\right\} \left(1 - \frac{f(t_1)}{g(t_1)}\right) g(t_1) \exp\left\{-\int_{t_1}^{t_0} \mathrm{d}t'\, g(t')\right\}$$

- For n intermediate rejections we obtain n nested integrals $\int_t^{t_0} \int_{t_1}^{t_0} \dots \int_{t_{n-1}}^{t_0}$
- Disentangling yields $1/n!$ and summing over all possible rejections gives

$$f(t) \exp\left\{-\int_t^{t_0} \mathrm{d}t'\, g(t')\right\} \sum_{n=0}^{\infty} \frac{1}{n!} \left[\int_t^{t_0} \mathrm{d}t' \left[g(t') - f(t')\right]\right]^n = f(t) \exp\left\{-\int_t^{t_0} \mathrm{d}t'\, f(t')\right\}$$

Monte-Carlo method for parton showers

- Start with set of n partons at scale t', which evolve collectively
  Sudakovs factorize, schematically

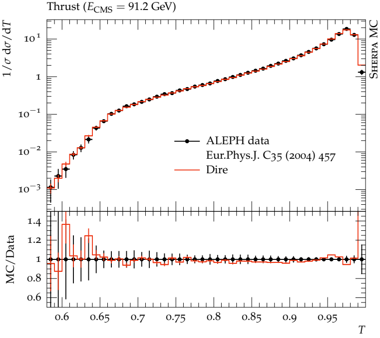
$$\Delta(t,t') = \prod_{i=1}^{n} \Delta_i(t,t')\;, \qquad\qquad \Delta_i(t,t') = \prod_{j=q,g} \Delta_{i\to j}(t,t')$$

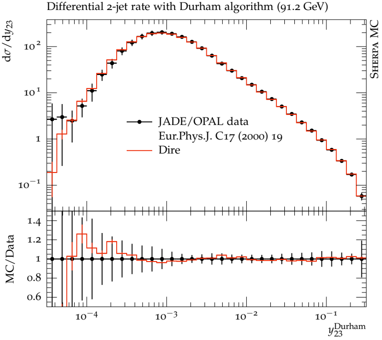
- Find new scale t where next branching occurs using veto algorithm
  - Generate t using overestimate $\alpha_s^{\max} P_{ab}^{\max}(z)$
  - Determine "winner" parton i and select new flavor j
  - Select splitting variable according to overestimate
  - Accept point with weight $\alpha_s(k_T^2)P_{ab}(z)/\alpha_s^{\max} P_{ab}^{\max}(z)$
- Construct splitting kinematics and update event record
- Continue until t falls below an IR cutoff

You will use this algorithm in the tutorial

Effects of the parton shower

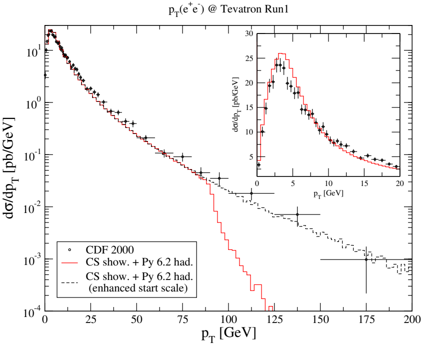
Effects of the parton shower





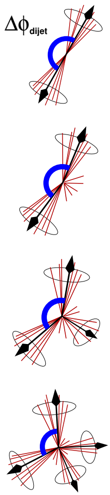
- Thrust and Durham $2 \to 3$-jet rate in $e^+e^- \to$hadrons
- Hadronization region to the right (left) in left (right) plot

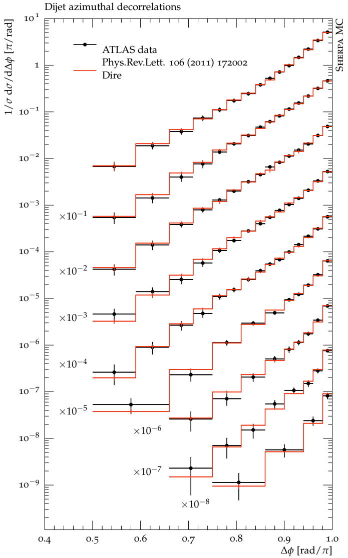
Effects of the parton shower

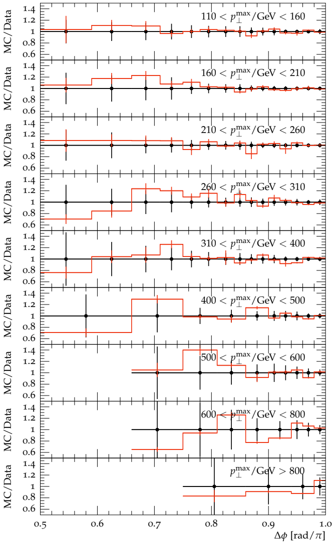


- Drell-Yan lepton pair production at Tevatron
- If hard cross section computed at leading order, then parton shower is only source of transverse momentum

Effects of the parton shower







Radiative corrections

The semi-classical picture

Semi-classical source theory

- Classical point charge on trajectory $y^\mu(s) \to$ conserved current $j^\mu(\mathrm{x})$

$$j^\mu(x) = g \int \mathrm{d}t \, \frac{dy^\mu(t)}{dt} \, \delta^{(4)}(x - y(t)) \, , \qquad\qquad g = \sqrt{4\pi\alpha}$$

- Fourier transform to momentum space

$$j^\mu(k) = \int \mathrm{d}^4x \, e^{ikx} \, j^\mu(x) = g \int \mathrm{d}t \, \frac{dy^\mu(t)}{dt} \, e^{iky(t)}$$

- Assume particle moves with momentum p_a if $t < 0$,
  is 'kicked' at origin $y^\mu(0) = 0$, and moves with p_b if $t > 0$

$$y^\mu(t) = t \, \frac{p^\mu(t)}{p_0(t)} = \begin{cases} t \, p_a^\mu / p_{a,0} & \text{if} \quad t < 0 \\ t \, p_b^\mu / p_{b,0} & \text{if} \quad t > 0 \end{cases}$$

- Introduce a regulator and Fourier transform ...

$$j^\mu(k) = g \int_{-\infty}^{0} \mathrm{d}t \, \frac{p_a^\mu}{p_{a,0}} \, \exp\left\{ i \left(\frac{p_a k}{p_{a,0}} - i\varepsilon \right) t \right\} + g \int_0^{+\infty} \mathrm{d}t \, \frac{p_b^\mu}{p_{b,0}} \, \exp\left\{ i \left(\frac{p_b k}{p_{b,0}} + i\varepsilon \right) t \right\}$$

Semi-classical source theory

- Classical current

$$j^\mu(k) = \; ig \left(\frac{p_b^\mu}{p_b k + i\varepsilon} - \frac{p_a^\mu}{p_a k - i\varepsilon} \right)$$

  - Spin independent
  - Conserved

- Now add the quantum part $\rightarrow$ current can create gauge bosons
  Interaction Hamiltonian density

$$\mathcal{H}_{\text{int}}(x) = j^\mu(x) A_\mu(x)$$

- Probability of no emission $\rightarrow$ vacuum persistence amplitude squared

$$|W_{a\to b}|^2 = |\langle 0| T \left[\exp\left\{ i \int \mathrm{d}^4x \, j^\mu(x) A_\mu(x) \right\} \right] |0\rangle|^2$$

- Can be expanded into power series

$$W_{a\to b} = \sum \frac{1}{n!} W_{a\to b}^{(n)} \,, \qquad\qquad W_{a\to b}^{(n)} \propto g^n$$

  - Zeroth order: $W_{a\to b}^{(0)} = 1$
  - First order: $\langle 0| A_\mu(x) |0\rangle = 0$

Semi-classical source theory

- Second order contribution

$$W^{(2)}_{a\to b} = -\int \mathrm{d}^4x \int \mathrm{d}^4y\, j^\mu(x) j^\nu(y) \langle 0|T\left[A_\mu(x)A_\nu(y)\right]|0\rangle$$
$$= -\int \mathrm{d}^4x \int \mathrm{d}^4y\, j^\mu(x)\, i\Delta_{F,\mu\nu}(x,y)\, j^\nu(y)$$

  - Emission of field quantum at x, propagation to y & absorption
  - Unobserved, i.e. a *virtual* correction

- Propagation described by time-ordered Green's function

$$i\Delta_F^{\mu\nu}(x,y) = \Theta(y_0 - x_0)\langle 0|A^\nu(y)A^\mu(x)|0\rangle + \Theta(x_0 - y_0)\langle 0|A^\mu(x)A^\nu(y)|0\rangle$$
$$= \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3\, 2E_k}\Big[\,\Theta(y_0 - x_0)e^{-ik(y-x)}$$
$$+\Theta(x_0 - y_0)e^{ik(y-x)}\Big] \sum_{\lambda=\pm} \varepsilon^\mu_\lambda(k,l)\varepsilon^{\nu\,*}_\lambda(k,l)$$
$$= -i\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{e^{-ik(y-x)}}{k^2 + i\varepsilon}\sum_{\lambda=\pm}\varepsilon^\mu_\lambda(k)\varepsilon^{\nu\,*}_\lambda(k)$$

Semi-classical source theory

- Insert into vacuum persistence amplitude

$$W^{(2)}_{a\to b} = -i\int \mathrm{d}^4x\int \mathrm{d}^4y\int \frac{\mathrm{d}^4k}{(2\pi)^4}\,\frac{e^{-ik(y-x)}}{k^2+i\varepsilon}\sum_{\lambda=\pm}\big(j(x)\varepsilon_\lambda(k)\big)\big(j(y)\varepsilon_\lambda(k)\big)^*$$

$$= -i\int \frac{\mathrm{d}^4k}{(2\pi)^4}\,\frac{1}{k^2+i\varepsilon}\sum_{\lambda=\pm}\big(j(k)\varepsilon_\lambda(k)\big)\big(j(k)\varepsilon_\lambda(k)\big)^*$$

- Use completeness relation for polarization vectors (e.g. axial gauge)

$$\sum_{\lambda=\pm}\varepsilon^\mu_\lambda(k,l)\,\varepsilon^{\nu\,*}_\lambda(k,l) = -g^{\mu\nu}+\frac{k^\mu l^\nu+k^\nu l^\mu}{kl}$$

- Complete second-order contribution $\big(p_a^2=p_b^2=0$, dim.reg., $\overline{\mathrm{MS}})$

$$W^{(2)}_{a\to b} = -i\,|g|^2\Big(\frac{\mu^2 e^{\gamma_E}}{4\pi}\Big)^\varepsilon\int\frac{\mathrm{d}^Dk}{(2\pi)^D}\,\frac{1}{k^2+i\varepsilon}\,\frac{2p_ap_b}{(p_ak)(p_bk)}$$

$$\xrightarrow{\text{IR only}} -\frac{\alpha}{\pi}\left(\frac{1}{\varepsilon^2}-\frac{1}{\varepsilon}\log\frac{2p_ap_b}{\mu^2}+\frac{1}{2}\log^2\frac{2p_ap_b}{\mu^2}-\frac{\pi^2}{12}+\mathcal{O}(\varepsilon)\right)$$

Exercise: Compute matrix element in first line from eqns above using the Landau gauge $-g^{\mu\nu}+\frac{k^\mu k^\nu}{k^2}$

Semi-classical source theory

- Real-emission contribution

$$\mathrm{d}W^2_{a\to bc}(p_c) = \frac{\mathrm{d}^3\vec{p}_c}{(2\pi)^3\,2E_c}\left|\langle\vec{p}_c|T\left[\exp\left\{i\int \mathrm{d}^4x\, j^\mu(x)A_\mu(x)\right\}\right]|0\rangle\right|^2 .$$

- Can be expanded into power series

$$\mathrm{d}W_{a\to bc}(p_c) = \sum \frac{1}{n!}\mathrm{d}W^{(n)}_{a\to bc}(p_c)\,, \qquad \mathrm{d}W^{(n)}_{a\to bc}(p_c) \propto g^n$$

  - Zeroth order: $\langle\vec{p}_c|0\rangle = 0$

- First-order term ($p_a^2 = p_b^2 = 0$, dim.reg., $\overline{\mathrm{MS}}$)

$$\begin{aligned}
\int \mathrm{d}W^{2\,(1)}_{a\to bc}(p_c) &= \int \frac{\mathrm{d}^3\vec{p}_c}{(2\pi)^3\,2E_c}\left|\,i\int \mathrm{d}^4x\, j^\mu(x)\langle\vec{p}_c|A_\mu(x)|0\rangle\right|^2 \\
&= -\int \frac{\mathrm{d}^3\vec{p}_c}{(2\pi)^3\,2E_c}\sum_{\lambda=\pm}\big(j(p_c)\varepsilon_\lambda(p_c)\big)\big(j(p_c)\varepsilon_\lambda(p_c)\big)^* \\
&\to |g|^2\left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^\varepsilon \int \frac{\mathrm{d}^D\vec{p}_c}{(2\pi)^D}\frac{2p_a p_b}{(p_a p_c)(p_b p_c)}\,\delta(p_c^2) \\
&\approx +\frac{\alpha}{\pi}\left(\frac{1}{\varepsilon^2} - \frac{1}{\varepsilon}\log\frac{2p_a p_b}{\mu^2} + \frac{1}{2}\log^2\frac{2p_a p_b}{\mu^2} - \frac{\pi^2}{12} + \mathcal{O}(\varepsilon)\right)
\end{aligned}$$

Semi-classical source theory

- So far we have

$$W^{(2)}_{a\to b} = -\frac{\alpha}{\pi}\left(\frac{1}{\varepsilon^2}-\frac{1}{\varepsilon}\log\frac{2p_ap_b}{\mu^2}+\frac{1}{2}\log^2\frac{2p_ap_b}{\mu^2}-\frac{\pi^2}{12}+\mathcal{O}(\varepsilon)\right)$$

$$\int \mathrm{d}W^{2\,(1)}_{a\to bc}(p_c) = +\frac{\alpha}{\pi}\left(\frac{1}{\varepsilon^2}-\frac{1}{\varepsilon}\log\frac{2p_ap_b}{\mu^2}+\frac{1}{2}\log^2\frac{2p_ap_b}{\mu^2}-\frac{\pi^2}{12}+\mathcal{O}(\varepsilon)\right)$$

  - Explicit form of unitarity condition (probability conservation)
  - Poles in ε cancel between virtual and real-emission correction
  - π^2 contributions due to D-dimensional phase space
- Double poles in ε only appear upon integration over loop momentum and full real-emission phase space $\rightarrow$ associated with unobserved region $\rightarrow$ can be removed explicitly (real-virtual cancelation)
- Remaining terms are double logarithms

$$W^{(2)}_{a\to b} \rightarrow -\frac{\alpha}{\pi}\left(\frac{1}{2}\log^2\frac{2p_ap_b}{\mu^2}-\frac{\pi^2}{12}+\mathcal{O}(\varepsilon)\right)$$

$$\int \mathrm{d}W^{2\,(1)}_{a\to bc}(p_c) \rightarrow +\frac{\alpha}{\pi}\left(\frac{1}{2}\log^2\frac{2p_ap_b}{\mu^2}-\frac{\pi^2}{12}+\mathcal{O}(\varepsilon)\right)$$

  - These terms survive if unitarity is broken by the measurement
    e.g. vetoed real radiation above a certain scale μ^2
    Exercise: Find more examples where real/virtual corrections are probed

Semi-classical source theory

- Order 2n contribution to vacuum persistence amplitude

$$W^{(2n)}_{a\to b} = \Big[\prod_{i=1}^{2n} i\int \mathrm{d}^4x_i j^{\mu_i}(x_i)\Big]\langle 0|T\Big[\prod_{i=1}^{2n} A_{\mu_i}(x_i)\Big]|0\rangle$$

- Decompose time-ordered product into Feynman propagators, use symmetry of integrand in currents

$$\begin{aligned}\frac{W^{(2n)}_{a\to b}}{(2n)!} &= \frac{(2n-1)(2n-3)\dots 3\cdot 1}{(2n)!}\Big[\prod_{i=1}^{2n} i\int \mathrm{d}^4x_i j^{\mu_i}(x_i)\Big] \\ &\qquad \times \prod_{i=1}^{n}\langle 0|T\left[A_{\mu_{2i}}(x_{2i})A_{\mu_{2i+1}}(x_{2i+1})\right]|0\rangle \\ &= \frac{1}{2^n n!}\left(-\int \mathrm{d}^4x\int \mathrm{d}^4y\, j^\mu(x) i\Delta_{\mu\nu}(x,y) j^\nu(y)\right)^n = \frac{1}{n!}\left(\frac{W^{(2)}_{a\to b}}{2}\right)^n .\end{aligned}$$

- Sum all orders in $\alpha \to$ vacuum persistence amplitude squared

$$|W_{a\to b}|^2 = \Big|\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{W^{(2)}_{a\to b}}{2}\right)^n\Big|^2 = \exp\left\{W^{(2)}_{a\to b}\right\} .$$

Semi-classical source theory – Summary

- Sudakov factor from first principles

  $$\Delta = |W_{a\to b}|^2 = \exp\left\{W^{(2)}_{a\to b}\right\}$$

  - Resummed virtual corrections at scale μ^2
  - Logarithmic structure same as real corrections

- For Abelian theories we can also use

  $$\Delta = \exp\left\{-\int \mathrm{d}W^{2\,(1)}_{a\to bc}\right\}$$

  - Agrees with heuristics based on probability conservation
  - Sufficient for most use cases in non-Abelian theories, but not exact
    Exercise: What is different in QCD?

- Universal, semi-classical integrand (Eikonal)

  $$\frac{2p_a p_b}{(p_a p_c)(p_b p_c)}$$

  - Leads to double logarithm $1/2\log^2(2p_a p_b/\mu^2)$
  - Originates in gauge boson radiation off conserved charge

Dipole radiation pattern

Geometric properties of semi-classical result

Structure of semi-classical matrix element

[Marchesini,Webber] NPB310(1988)461

- Matrix element can be written in terms of energies and angles

  $$\frac{2p_a p_b}{(p_a p_c)(p_c p_b)} = \frac{W_{ab,c}}{E_c^2}$$

  Angular "radiator" function

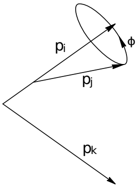
  $$W_{ab,c} = \frac{1 - \cos\theta_{ab}}{(1 - \cos\theta_{ac})(1 - \cos\theta_{bc})}$$

- Divergent as $\theta_{ac} \to 0$ and as $\theta_{bc} \to 0$
  $\to$ Expose individual singularities using $W_{ab,c} = \tilde{W}^a_{ab,c} + \tilde{W}^b_{ba,c}$

  $$\tilde{W}^a_{ab,c} = \frac{1}{2}\left[\frac{1 - \cos\theta_{ab}}{(1 - \cos\theta_{ac})(1 - \cos\theta_{bc})} + \frac{1}{1 - \cos\theta_{ac}} - \frac{1}{1 - \cos\theta_{bc}}\right]$$

  - Divergent as $\theta_{ac} \to 0$, but regular as $\theta_{bc} \to 0$
  - Convenient properties upon integration over azimuthal angle

Structure of semi-classical matrix element



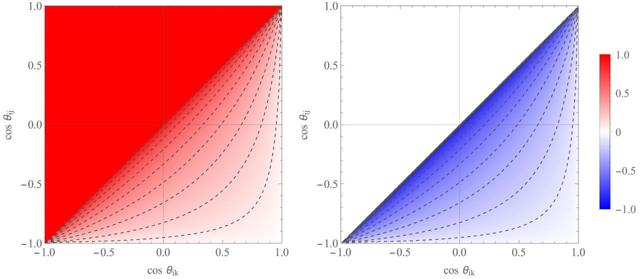
- Work in a frame where direction of $\vec{p}_a$ aligned with z-axis

$$\cos\theta_{bc} = \cos\theta_b \cos\theta_c + \sin\theta_b \sin\theta_c \cos\phi_c$$

- Integration over ϕ_c yields

$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi_c \tilde{W}^a_{ab,c} = \frac{1}{1-\cos\theta_c} \times \begin{cases} 1 & \text{if} \quad \theta_c < \theta_b \\ 0 & \text{else} \end{cases}$$

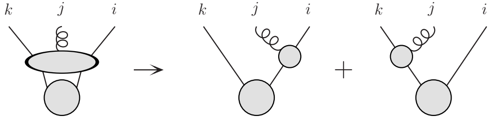
  - On average, no radiation outside cone defined by parent dipole
  - Differential radiation pattern more intricate:
    Positive & negative contributions outside cone sum to zero



Structure of semi-classical matrix element

- Alternative approach: partial fraction matrix element & match to collinear sectors [Ellis,Ross,Terrano] NPB178(1981)421, [Catani,Seymour] hep-ph/9605323

$$\frac{p_i p_k}{(p_i p_j)(p_j p_k)} \to \frac{1}{p_i p_j}\frac{p_i p_k}{(p_i+p_k)p_j} + \frac{1}{p_k p_j}\frac{p_i p_k}{(p_i+p_k)p_j}$$



  - Convenient, Lorentz invariant formulation
  - Easy to integrate and use in NLO IR subtraction
  - Captures matrix element both in angular ordered and unordered region

Seemingly ideal formulation of antenna radiation

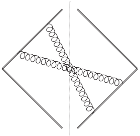
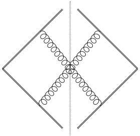
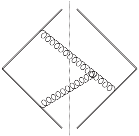
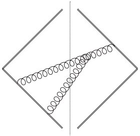
But theory still Abelian, so let's move on ...

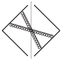
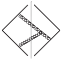
Approaching realistic QCD

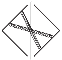
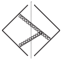
Structure of non-Abelian result

Explicit example – 2-gluon emission

- Semi-classical matrix element squared for $q(i)\bar{q}(j)g(1)g(2)$

 $+$  $+$  $+$  $+$ swaps

- Color factors
  -  $\propto \mathrm{Tr}\left[T^a T^b T^a T^b\right] = -C_F\left(\frac{C_A}{2} - C_F\right)$
  -  $\propto F^c_{ab}\mathrm{Tr}\left[T^a T^b T^c\right] = C_F\frac{C_A}{2}$

  Exercise: Where does the structure of the first term come from?
- Kinematical factors
  -  $\propto \frac{p_i p_j}{(p_i p_1)(p_1 p_j)}\frac{p_i p_j}{(p_i p_2)(p_2 p_j)}$
  -  $\propto \frac{p_i p_j}{(p_i p_1)(p_1 p_j)}\frac{p_i p_1}{(p_i p_2)(p_2 p_1)}$

  Exercise: Can you derive them?

Explicit example – 2-gluon emission

- Complete matrix element (Note: $s_{ij} = 2p_ip_j$)

$$C_F \frac{s_{ij}}{s_{i1}s_{j1}} \left(\frac{C_A}{2} \left(\frac{s_{i1}}{s_{i2}s_{12}} + \frac{s_{j1}}{s_{j2}s_{12}} \right) + \left(C_F - \frac{C_A}{2} \right) \frac{s_{ij}}{s_{i2}s_{j2}} \right)$$

  - Factorizes into first and second emission contribution
  - Non-Abelian color factors mix with Abelian kinematics
- Two important limits
  - $N_c \to \infty$, $C_A =$ const (large N_c limit):

$$\left(\frac{C_A}{2} \right)^2 \left(\frac{s_{ij}}{s_{i2}s_{12}s_{j1}} + \frac{s_{ij}}{s_{i1}s_{12}s_{j2}} \right)$$

  - $N_c \to 0$, $C_F =$ const (Abelian limit):

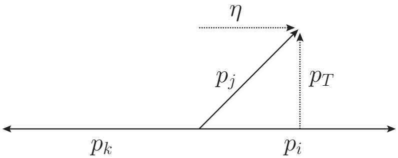
$$C_F^2 \frac{s_{ij}}{s_{i1}s_{j1}} \frac{s_{ij}}{s_{i2}s_{j2}}$$

Nice and simple formulae, but what have we learned?

Need a tool to visualize what's happening

Making sense of things – The Lund plane

- Compute everything in center-of-mass frame of quarks



- Write momenta in Sudakov decomposition

$$p_1 = p_1^+ + p_1^- + p_{T,1}$$

  - On-shell condition: $p_1^2 = 2(p_1^+ p_1^- - p_{T,1}^2)$
  - "$-$"-projection: $p_1^- = 2p_i p_1 / \sqrt{2p_i p_j}$
  - "$+$"-projection: $p_1^+ = 2p_j p_1 / \sqrt{2p_i p_j}$

- Simple expressions for transverse momentum and rapidity
  - $p_{T,1}^2 = \dfrac{2(p_i p_1)(p_j p_1)}{p_i p_j}$
  - $\eta_1 = \dfrac{1}{2} \ln \dfrac{p_i p_1}{p_j p_1}$
- Semi-classical abelian matrix element squared $\propto 1/p_T^2$

Making sense of things – The Lund plane

- Rewrite rapidity using transverse momentum

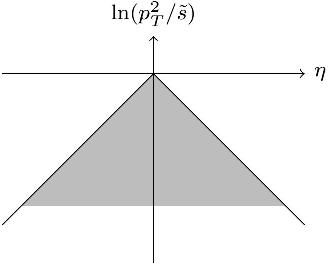
$$\eta_1 = \frac{1}{2}\ln\frac{p_i p_1}{p_j p_1} = \frac{1}{2}\ln\frac{s_{i1}^2}{p_{T,1}^2 s_{ij}} = \frac{1}{2}\ln\frac{p_{T,1}^2 s_{ij}}{s_{j1}^2}$$

- In momentum conserving parton branching $(\tilde{p}_i, \tilde{p}_j) \to (p_i, p_j, p_1)$

$$-\frac{1}{2}\ln\frac{\tilde{s}_{ij}}{p_{T,1}^2} \le \eta_1 \le \frac{1}{2}\ln\frac{\tilde{s}_{ij}}{p_{T,1}^2}$$

- Differential phase-space element $\propto \mathrm{d}p_T^2\,\mathrm{d}\eta$ (exercise)
- The Lund plane
  - $\eta, \ln(p_T^2/\tilde{s})$ plane
  - Phase space bounded by diagonals
  - Single-emission semi-classical radiation probability a constant

  Exercise: How do the double logarithms in the integrated matrix element emerge in the Lund plane?

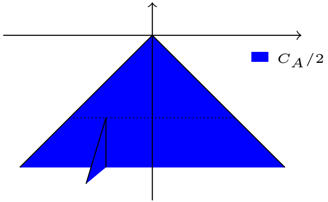


Explicit example – 2-gluon emission

- Limits of 2-gluon matrix element in Lund coordinates
  - $N_c \to \infty$, $C_A =$ const (large N_c limit):

    $$\frac{(C_A/2)^2}{p_{T,1}^{2\,(i,j)}\, p_{T,2}^{2\,(i,1)}} + \left(i \leftrightarrow j\right)$$

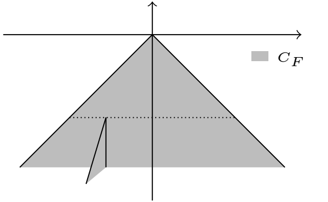
    - Gray area - C_F
    - Blue area - $C_A/2$



  - $N_c \to 0$, $C_F =$ const (Abelian limit):

    $$\frac{C_F^2}{p_{T,1}^{2\,(i,j)}\, p_{T,2}^{2\,(i,j)}}$$

    - Gray area - C_F



Explicit example – 2-gluon emission

- Full 2-guon matrix element

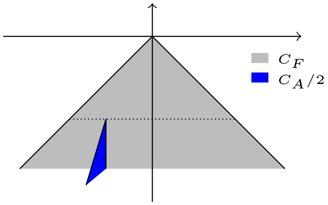
$$\frac{C_F}{p_{T,1}^2}\frac{1}{E_2^2}\left(\frac{C_A}{2}\left(\tilde{W}^i_{i1,2}+\tilde{W}^1_{i1,2}-\tilde{W}^i_{ij,2}\right)+C_F\tilde{W}^i_{ij,2}+\left(i\leftrightarrow j\right)\right)$$

- Rewrite using single-soft radiator $\bar{W}^{1,j}_{i,2}=\tilde{W}^i_{i1,2}-\tilde{W}^i_{ij,2}$

$$\frac{C_F}{p_{T,1}^2}\frac{1}{E_2^2}\left(\frac{C_A}{2}\left(\bar{W}^{1,j}_{i,2}+\tilde{W}^1_{i1,2}\right)+C_F\tilde{W}^i_{ij,2}+\left(i\leftrightarrow j\right)\right)$$

  - Azimuthally integrated $\bar{W}^{1,j}_{i,2}$ vanishes if $\theta_{i2}<\min(\theta_{i1},\theta_{ij})$
  - Azimuthally integrated $\tilde{W}^1_{i1,2}$ vanishes if $\theta_{12}>\theta_{i1}$

- For $\theta_{j1}\ll\theta_{ij}$ and $\theta_{12}>\theta_{i1}$, both $C_A/2$ terms vanish $\rightarrow$ Radiation from C_F term alone



The simplest manifestation of angular ordering in QCD

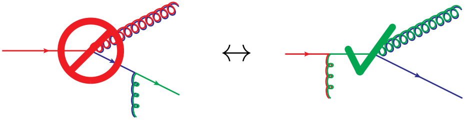
Color coherence and angular ordering

The heuristic picture

Color coherence and the dipole picture

[Marchesini,Webber] NPB310(1988)461
[Gustafsson,Pettersson] NPB306(1988)746

- Individual color charges inside a color dipole cannot be resolved if gluon wavelength larger than dipole size $\rightarrow$ emission off "mother"



- Net effect is destructive interference outside a cone with opening angle set by emitting color dipole
- Known in QED as the Chudakov effect

Let's have a look at the implementation

The phase-space integrals

Phase-space factorization

- Differential n-particle phase space element (massless partons)

$$\mathrm{d}\Phi_n(p_1,\ldots,p_n;P) = \left[\prod_{i=1}^{n} \frac{\mathrm{d}^4 p_i}{(2\pi)^3}\delta(p_i^2)\right](2\pi)^4\delta^{(4)}\Big(P-\sum_i p_i\Big)$$

  - Obeys s-channel factorization formula [Byckling,Kajantie] NPB9(1969)568
- Use factorization to split off a $1 \to 2$ decay

$$\mathrm{d}\Phi_n(p_1,\ldots,p_n;P) = \mathrm{d}\Phi_{n-1}(P_{12},p_3,\ldots,p_n;P)\,\frac{\mathrm{d}P_{12}^2}{2\pi}\,\mathrm{d}\Phi_2(p_1,p_2;P_{12})$$

- 2-body phase space in center-of-mass frame of light-like p_1 & p_2

$$\mathrm{d}\Phi_2(p_1,p_2;P) = \frac{1}{32\pi^2}\,\mathrm{d}\cos\theta\,\mathrm{d}\phi$$

- Rewrite in terms of light-cone momentum fraction $z = (1+\cos\theta)/2$

$$\mathrm{d}\Phi_n(p_1,\ldots,p_n;P) = \mathrm{d}\Phi_{n-1}(P_{12},p_3,\ldots,p_n;P)\,\frac{1}{16\pi^2}\,\mathrm{d}s_{12}\,\mathrm{d}z\,\frac{\mathrm{d}\phi}{2\pi}$$

  - Most parton showers evolve on-shell states into on-shell states
  - Must redefine $P_{12} \to \tilde{P}_{12}$, where $\tilde{P}_{12}^2 = 0$, while P^2=const

How the redefinition is achieved is to some extent arbitrary

This is referred to as the "recoil scheme"

Putting everything together

I – Angular ordered evolution

Angular ordered parton showers

- Matrix element

$$|M|^2 = |g^2|\,\frac{2p_ap_b}{(p_ap_c)(p_bp_c)} + \text{spin dependent terms}$$

  - Define splitting function $2P_{ac} = 2(p_ap_c)\,|M|^2$

- Differential phase space

$$\mathrm{d}\Phi_{+1} \approx \frac{1}{16\pi^2}\,\mathrm{d}s_{ac}\,\mathrm{d}z\,\frac{\mathrm{d}\phi}{2\pi}$$

  - Rewrite $z = \dfrac{1+\cos\theta_{ab}}{2} = \dfrac{p_ap_b}{(p_a+p_c)p_b}$

- Differential radiation probability

$$\mathrm{d}\mathcal{P} = \mathrm{d}\Phi_{+1}|M|^2 \approx \frac{\mathrm{d}(p_ap_c)}{(p_ap_c)}\,\mathrm{d}z\,\frac{\alpha_s}{2\pi}\,P_{ac}(z) = \frac{\mathrm{d}\tilde{q}^2}{\tilde{q}^2}\,\mathrm{d}z\,\frac{\alpha_s}{2\pi}\,P_{ac}(z)$$

  - Semi-classical splitting function $P_{ac}(z) = 2C_a\,\dfrac{z}{1-z}$

    Add spin-dependent terms for complete result in collinear limit

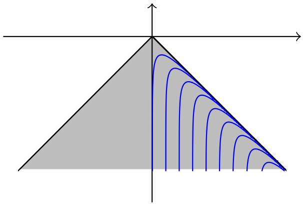
  - Ordering parameter $\tilde{q}^2 = \dfrac{2p_ap_c}{z(1-z)} \approx 4E_{ac}^2\sin^2\dfrac{\theta_{ac}}{2}$

Angular ordered parton showers

- Differential radiation probability

$$\mathrm{d}\mathcal{P} = \mathrm{d}\Phi_{+1}|M|^2 = \frac{\mathrm{d}\tilde{q}^2}{\tilde{q}^2}\,\mathrm{d}z\,\frac{\alpha_s}{2\pi}\,P_{ac}(z)$$

  - Dipole radiation becomes monopole radiation
    $\rightarrow$ parton (not *dipole*) shower
  - Non-Abelian structure of QCD simplifies
    $\rightarrow$ radiation off mean charge C_F or C_A
- Lund plane filled from center to edges
  - Random walk in p_T^2
  - Color factors correct for observables insensitive to azimuthal correlations
  - Small dead zone at $\ln(p_T^2/\tilde{s}) \approx 0$



Putting everything together

II – Dipole evolution

Dipole showers

- Matrix element

$$|M|^2 = |g^2| \frac{2p_a p_b}{(p_a p_c)(p_b p_c)} + \text{spin dependent terms}$$

  - Define splitting function $P_{ac} = p_{T,c}^2 |M|^2$
- Differential phase space

$$\mathrm{d}\Phi_{+1} \approx \frac{1}{16\pi^2}\, \mathrm{d}s_{ac}\, \mathrm{d}z\, \frac{\mathrm{d}\phi}{2\pi}$$

  - Rewrite $z = 1 - \frac{s_{ac}}{\tilde{s} - s_{ac}}\, \mathrm{e}^{-2\eta_c}$
- Differential radiation probability for the dipole

$$\mathrm{d}\mathcal{P} = \mathrm{d}\Phi_{+1} |M|^2 \approx \frac{\mathrm{d}p_{T,c}^2}{p_{T,c}^2}\, \mathrm{d}\eta_c\, \frac{\alpha_s}{2\pi}\, \tilde{P}_{ac}(z)$$

  - Semi-classical splitting function $\tilde{P}_{ac}(z) = 2C_a$

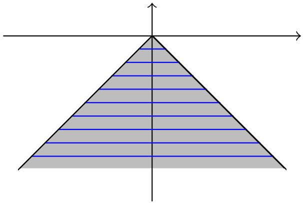
    Add spin-dependent terms for complete result in collinear limit
  - Ordering parameter $p_{T,c}^2$

Dipole showers

- Differential radiation probability for the dipole

$$\mathrm{d}\mathcal{P} = \mathrm{d}\Phi_{+1}|M|^2 \approx \frac{\mathrm{d}p_{T,c}^2}{p_{T,c}^2}\,\mathrm{d}\eta_c\,\frac{\alpha_s}{2\pi}\,\tilde{P}_{ac}(z)$$

  - Semi-classical dipole radiation has constant probability
  - Due to ordering in $p_{T,c}^2$ no natural way to recover correct color factors ($\nearrow$ later)
- Lund plane filled from top to bottom
  - Random walk in η
  - Color factors in improved leading color approximation
  - Both ends of dipole evolve simultaneously
  - No dead zones



Putting everything together

III – Dipole-like evolution

Dipole-like showers

- Matrix element

$$|M|^2 = |g^2| \frac{2p_a p_b}{(p_a p_c)(p_b p_c)} + \text{spin dependent terms}$$

  - Partial fraction $|M|^2 = |g^2| \frac{1}{p_a p_c} \frac{2p_a p_b}{(p_a + p_b)p_c} + \Big(a \leftrightarrow b\Big)$
  - Define splitting function $2P_{ac} = 2|g^2| \frac{2p_a p_b}{(p_a + p_b)p_c}$

- Differential phase space

$$\mathrm{d}\Phi_{+1} \approx \frac{1}{16\pi^2} \, \mathrm{d}s_{ac} \, \mathrm{d}z \, \frac{\mathrm{d}\phi}{2\pi}$$

  - Rewrite $z = \frac{1 + \cos\theta_{ab}}{2} = \frac{p_a p_b}{(p_a + p_c)p_b}$

- Differential radiation probability

$$\mathrm{d}\mathcal{P} = \mathrm{d}\Phi_{+1}|M|^2 \approx \frac{\mathrm{d}p_{T,c}^2}{p_{T,c}^2} \, \mathrm{d}z \, \frac{\alpha_s}{2\pi} \, \bar{P}_{ac}(z)$$

  - Semi-classical splitting function $\bar{P}_{ac}(z) = 2C_a \left(\frac{1-z}{(1-z)^2 + p_{T,c}^2/\tilde{s}} - 1 \right)$

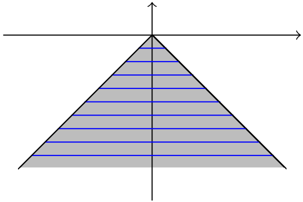
    Add spin-dependent terms for complete result in collinear limit
  - Ordering parameter $p_{T,c}^2$

Dipole-like showers

- Differential radiation probability

$$\mathrm{d}\mathcal{P} = \mathrm{d}\Phi_{+1}|M|^2 \approx \frac{\mathrm{d}p_{T,c}^2}{p_{T,c}^2}\,\mathrm{d}z\,\frac{\alpha_s}{2\pi}\,\bar{P}_{ac}(z)$$

  - Unified picture of parton and dipole evolution
  - Due to ordering in $p_{T,c}^2$ no natural way to recover correct color factors ($\nearrow$ later)
- Lund plane filled from top to bottom
  - Random walk in η
  - Color factors in improved leading color approximation
  - No dead zones

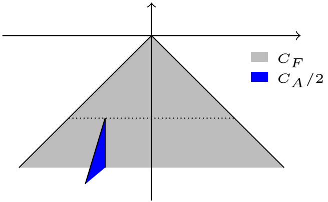


How to color the Lund plane

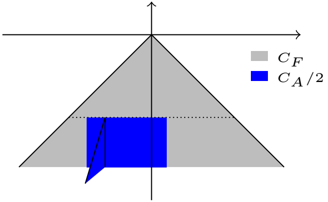
Multiple emission pattern of showers

Radiation pattern of angular ordered and dipole showers

- In angular ordered showers angles are measured in the event center-of-mass frame $\to$ coherence effects modeled by angular ordering variable agree on average with matrix element



- In dipole-like showers angles effectively measured in center-of-mass frame of emitting color dipole $\to$ angular coherence not reflected by setting average QCD charge

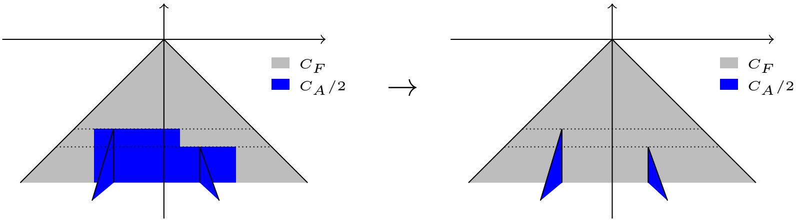


- Emission off "back plane" in Lund diagram should be associated with C_F, but is partly associated with $C_A/2$ in dipole showers
- All-orders problem that appears first in 2-gluon emission case

Correcting the radiation pattern of dipole showers

[Gustafsson] NPB392(1993)251

- Analyze rapidity of gluon emission in event center-of-mass frame
- Sectorize phase space and assign gluon to closest parton
  $\rightarrow$ choose corresponding color charge for evolution
- Same technology for higher number of emissions



- Starting with 4 emissions, there be "color monsters"
  - Quartic Casimir operators (easy)
  - Non-factorizable contributions (hard)

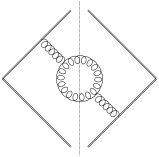
Not captured in either angular ordered or corrected dipole evolution

Universal higher-order corrections

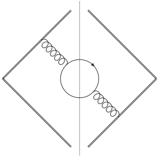
The CMW scheme

Soft-collinear enhanced terms at NLO

- Approximate soft-gluon emission times collinear decay in $q(i)\bar{q}(j)g(1)g(2)$ using semi-classical limit and gluon splitting function



$$+$$



$$= \sum_{b=q,g} j_{ij,\mu}(p_{12}) j_{ij,\nu}(p_{12}) \, \frac{P^{\mu\nu}_{gb}(z_1)}{s_{12}}$$

$$P^{\mu\nu}_{gq}(z) = T_R \left(-g^{\mu\nu} + 4\, z(1-z)\, \frac{k_\perp^\mu k_\perp^\nu}{k_\perp^2} \right)$$

$$P^{\mu\nu}_{gg}(z) = C_A \left(-g^{\mu\nu} \left(\frac{z}{1-z} + \frac{1-z}{z} \right) - 2\,(1-\varepsilon) z(1-z)\, \frac{k_\perp^\mu k_\perp^\nu}{k_\perp^2} \right)$$

- Combine with phase space for one parton emission in collinear limit $D = 4 - 2\varepsilon$, $y = s_{12}/Q^2$, see for example [Catani,Seymour] hep-ph/9605323

$$\mathrm{d}\Phi_{+1} = \frac{Q^{2-2\varepsilon}}{16\pi^2} \frac{(4\pi)^\varepsilon}{\Gamma(1-\varepsilon)} \, \mathrm{d}y \, \mathrm{d}z \, \left[y\, z(1-z) \right]^{-\varepsilon}$$

- Perform Laurent series expansion

$$\frac{1}{y^{1+\varepsilon}} = -\frac{\delta(y)}{\varepsilon} + \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} \left(\frac{\ln^n y}{y} \right)_+$$

Soft-collinear enhanced terms at NLO

- $\mathcal{O}(\varepsilon^0)$ remainder terms proportional to

$$g \to q\bar{q}: \quad T_R \Big[2z(1-z) + \big(1 - 2z(1-z)\big) \ln(z(1-z)) \Big]$$

$$g \to gg: \quad 2C_A \left[\frac{\ln z}{1-z} + \frac{\ln(1-z)}{z} + \big(-2 + z(1-z)\big) \ln(z(1-z)) \right]$$

- Integration over z gives

$$\left(\frac{67}{18} - \frac{\pi^2}{3} \right) C_A - \frac{10}{9} T_R n_f$$

  - Some additional terms from semi-classical diagrams
  - Contribution from exact virtual correction (no unitarity!)
  - Only π^2 term changed (identical to $\mathcal{N} = 4$ SYM)

- Sums to two-loop cusp anomalous dimension

$$K = \left(\frac{67}{18} - \frac{\pi^2}{6} \right) C_A - \frac{10}{9} T_R n_f$$

  - Local K-factor for soft-gluon emission
  - Scheme dependent: originates in dim. reg. and $\overline{\mathrm{MS}}$

K can be absorbed into an effective coupling

This is called the CMW scheme [Catani,Marchesini,Webber] NPB349(1991)635

Connection to analytic resummation

Event shapes at NLL accuracy

How to assess formal precision?

- Angular ordered parton showers are proven to be NLL accurate for certain observables, provided that the CMW scheme is used
- But how do we quantify this for other showers?
  Can we establish a limit where parton showers should reproduce NLL exactly?
- Let's use a well-established result as an example
  - Observable: Thrust in $e^+e^- \to$hadrons
  - Method: Caesar [Banfi,Salam,Zanderighi] hep-ph/0407286

This discussion will be quite technical, so why have it at all?

Because the relevant limit is the $\alpha_s \to 0$ limit.

Sounds pretty unphysical, so it's definitely worth a closer look!

NLL resummation for simple additive observables

- Contribution of one emission with momentum k to "thrust" $v = 1 - T$

$$V(k) = \left(\frac{k_T}{Q}\right) e^{-\eta} \qquad \to \qquad V(\{p\},\{k\}) = \sum_i V(k_i)$$

where k_T, $\eta = \log((1-z)Q/k_T) \to$ Lund coordinates of soft-gluon momentum

- Define a shower evolution variable $\xi = k_T^2/(1-z)$
- Integrated one-emission probability for $\xi > Q^2 v$

$$R_{\mathrm{PS}}(v) = 2\int_{Q^2 v}^{Q^2} \frac{d\xi}{\xi} \int_{z_{\min}}^{z_{\max}} dz \, \frac{\alpha_s\left(k_T^2\right)}{2\pi} C_F \left[\frac{2}{1-z} - (1+z)\right] \Theta(\eta)$$

z-limits from momentum conservation, $\Theta(\eta)$ implements angular ordering

- Approximate to NLL accuracy

$$R_{\mathrm{NLL}}(v) = 2\int_{Q^2 v}^{Q^2} \frac{d\xi}{\xi} \left[\int_0^1 dz \, \frac{\alpha_s\left(k_T^2\right)}{2\pi} \frac{2\,C_F}{1-z} \Theta(\eta) - \frac{\alpha_s(\xi)}{\pi} C_F B_q\right]$$

Exercise: Can you derive the value of B_q?

Origin of the $\alpha_s \to 0$ limit – The $\mathcal{F}$ function

- Define the cumulative cross section $\Sigma(v)$

$$\Sigma\left(v\right) = e^{-R(v)}\mathcal{F}\left(v\right)$$

- Obtained from the all-orders resummed result

$$\Sigma(v) = \int \mathrm{d}^3 k_1 |M(k_1)|^2 \exp\left\{-\int_{\varepsilon v_1} \mathrm{d}^3 k |M(k)|^2\right\}$$
$$\times \sum_{m=0}^{\infty} \frac{1}{m!}\left(\prod_{i=2}^{m+1}\int_{\varepsilon v_1}^{v_1} \mathrm{d}^3 k_i |M(k_i)|^2\right)\Theta\Big(v - V(\{p\}, k_1, \ldots, k_n)\Big)$$

by Taylor expansion of virtual corrections in ε

$$\exp\left\{-\int_{\varepsilon v_1} \mathrm{d}^3 k |M(k)|^2\right\} = e^{-R(v)}\, e^{-R' \ln \frac{v}{\varepsilon v_1}}$$

- Definition of $\mathcal{F}(v)$

$$\mathcal{F}(v) = \int \mathrm{d}^3 k_1 |M(k_1)|^2 \, e^{-R' \ln \frac{v}{\varepsilon v_1}} \sum_{m=0}^{\infty} \frac{1}{m!}\left(\prod_{i=2}^{m+1}\int_{\varepsilon v_1}^{v_1} \mathrm{d}^3 k_i |M(k_i)|^2\right)$$
$$\times \Theta\Big(v - V(\{p\}, k_1, \ldots, k_n)\Big)$$

  - Purely NLL (no leading logarithms!)
  - Accounts for multiple-emission effects

Origin of the $\alpha_s \to 0$ limit – The $\mathcal{F}$ function

- In order to make this calculable, make the following approximations
  - Observable is recursively infrared and collinear safe
    $\to$ Can scale phase space $\int_{\varepsilon v_1}^{v_1} \to \int_{\varepsilon v}^{v}$
  - **Hold $\alpha_s(Q^2)\ln v$ fixed, while taking the limit $v \to 0$**
    $\to$ Can factorize integrals and neglect kinematic edge effects
- Reduces $\mathcal{F}$-function to convenient form

$$\mathcal{F}\left(v\right) = e^{R'(v)\ln\epsilon} \sum_{m=0}^{\infty} \frac{1}{m!} \left(\prod_{i=1}^{m} R'(v) \int_{\epsilon}^{1} \frac{d\zeta_i}{\zeta_i} \right) \Theta\Big(1 - \sum_{j=1}^{m} \zeta_j\Big)$$

  - For thrust and similar observales, $\mathcal{F}(v) = \dfrac{e^{-\gamma_E R'}}{\Gamma(1+R')}$

Remarkably simple and clean (no NNLL contamination)

Could only be achieved because of the limit $v \to 0$ / $\alpha_s \to 0$

$\alpha_s \to 0$ benchmark tests *exactly* NLL, nothing less or more

Differences between pure NLL and parton shower

[Reichelt,Siegert,SH] arXiv:1711.03497

- Schematic difference between analytic resummation and parton shower
  - $\Sigma_{\rm NLL}(v)$ determined at exactly NLL
  - $\Sigma_{\rm PS}(v)$ determined by unitarity
- One can find a unified NLL/PS expression for $R(v)$ and $\Sigma(v)$

$$\Sigma\,(v) = \exp\left\{-\int_v \frac{d\xi}{\xi} R'_{>v}(\xi) - \int_{v_{\rm min}}^{v} \frac{d\xi}{\xi} R'_{<v}(\xi)\right\}$$
$$\times \sum_{m=0}^{\infty} \frac{1}{m!} \left(\prod_{i=1}^{m} \int_{v_{\rm min}} \frac{d\xi_i}{\xi_i} R'_{<v}(\xi_i)\right) \Theta\left(v - \sum_{j=1}^{m} V(\xi_i)\right)$$

where

$$R'_{\lessgtr v}(\xi) = \frac{\alpha_s^{\lessgtr v,\rm soft}(\mu^2_{\lessgtr})}{\pi} \int_{z^{\rm min}}^{z^{\rm max}_{\lessgtr v,\rm soft}} dz\, \frac{C_{\rm F}}{1-z} - \frac{\alpha_s^{\lessgtr v,\rm coll}(\mu^2_{\lessgtr v})}{\pi} \int_{z^{\rm min}}^{z^{\rm max}_{\lessgtr v,\rm coll}} dz\, C_{\rm F} \frac{1+z}{2}$$

Differences between pure NLL and parton shower

- Isolated differences in terms of resolved/unresolved splitting probability:

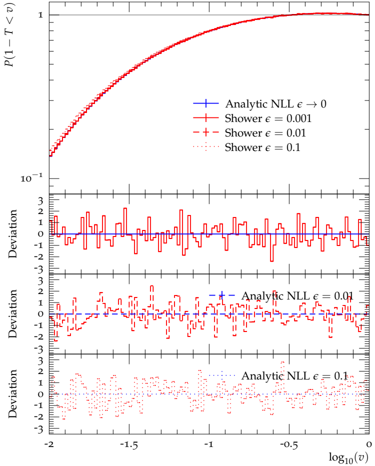
$$R'_{\lessgtr v}(\xi) = \frac{\alpha_s^{\lessgtr v,\mathrm{soft}}(\mu^2_{\lessgtr})}{\pi}\int_{z^{\min}}^{z^{\max}_{\lessgtr v,\mathrm{soft}}} dz\,\frac{C_\mathrm{F}}{1-z} - \frac{\alpha_s^{\lessgtr v,\mathrm{coll}}(\mu^2_{\lessgtr v})}{\pi}\int_{z^{\min}}^{z^{\max}_{\lessgtr v,\mathrm{coll}}} dz\,C_\mathrm{F}\,\frac{1+z}{2}$$

	NLL	Parton Shower
$z^{\max}_{>v,\mathrm{soft}}$	$1-(\xi/Q^2)^{\frac{a+b}{2a}}$	
$\mu^2_{>v,\mathrm{soft}}$	$\xi(1-z)^{\frac{2b}{a+b}}$	
$\alpha_s^{>v,\mathrm{soft}}$	2-loop CMW	
$z^{\max}_{<v,\mathrm{soft}}$	$1-v^{\frac{1}{a}}$	$1-(\xi/Q^2)^{\frac{a+b}{2a}}$
$\mu^2_{<v,\mathrm{soft}}$	$Q^2 v^{\frac{2}{a+b}}(1-z)^{\frac{2b}{a+b}}$	$\xi(1-z)^{\frac{2b}{a+b}}$
$\alpha_s^{<v,\mathrm{soft}}$	1-loop	2-loop CMW

	NLL	Parton Shower
$z^{\max}_{>v,\mathrm{coll}}$	1	$1-(\xi/Q^2)^{\frac{a+b}{2a}}$
$\mu^2_{>v,\mathrm{coll}}$	ξ	$\xi(1-z)^{\frac{2b}{a+b}}$
$\alpha_s^{>v,\mathrm{coll}}$	1-loop	2-loop CMW
$z^{\max}_{<v,\mathrm{coll}}$	0	$1-(\xi/Q^2)^{\frac{a+b}{2a}}$
$\mu^2_{<v,\mathrm{coll}}$	n.a.	$\xi(1-z)^{\frac{2b}{a+b}}$
$\alpha_s^{<v,\mathrm{coll}}$	n.a.	2-loop CMW

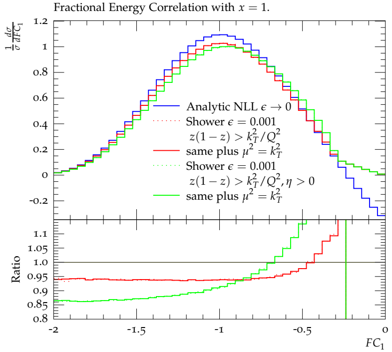
- Can cast pure NLL into PS language by using NLL expressions in PS
- Can study each effect in detail by reverting changes back to PS

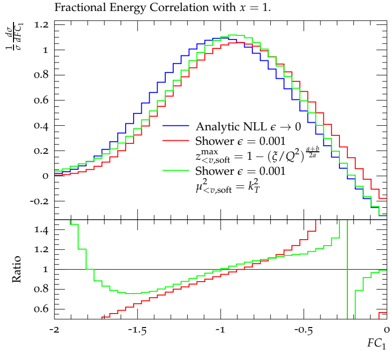
Implementing NLL resummation as a shower



- Modified parton shower exactly reproduces pure NLL result
- $E_{\rm cms}$=91.2 GeV, $\alpha_s(M_Z) = 0.118$ fixed flavor $n_f = 5$

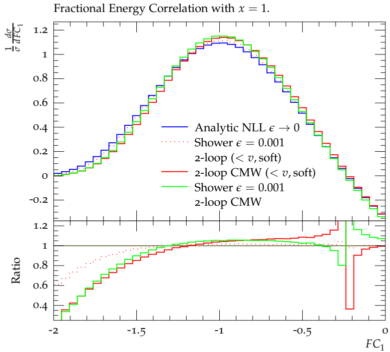
Local four momentum conservation and unitarity

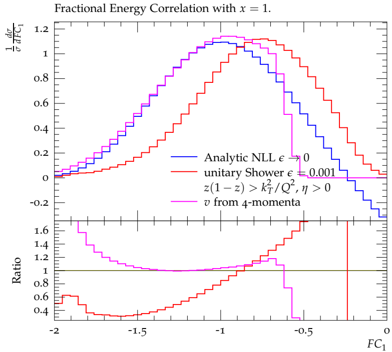




- NLL→PS in $z_{\min/\max}$
  (4-momentum conservation)
- NLL→PS in $z^{\max}_{>v,\mathrm{coll}}$
  (phase-space sectorization)
- NLL→PS in $\mu^2_{>v,\mathrm{coll}}$
  (conventional)
- NLL→PS in $z^{\max}_{<v,\mathrm{soft}}$
  (from PS unitarity)
- NLL→PS in $\mu^2_{<v,\mathrm{soft}}$
  (from PS unitarity)

Running coupling and global momentum conservation





- NLL→PS in 2-loop CMW $< v, \mathrm{soft}$ (from PS unitarity)
- NLL→PS in 2-loop CMW overall (conventional)
- NLL→PS in observable (use experimental definition)

Overall assessment

- Simplest process and simplest observable,
  still sizable differences away from $v \to 0$ limit
- Due to kinematic edge effects & unitarity
- At NLL, none of the methods is formally better
  $\to$ Difference is a true systematic uncertainty

The $\alpha_s \to 0$ limit is mandatory for exact comparison

Away from this limit there are important systematic effects

Problems with recoil

Correcting the momentum mapping

Momentum mapping in dipole-like showers

[Dasgupta,Dreyer,Hamilton,Monni,Salam] arXiv:1805.09327

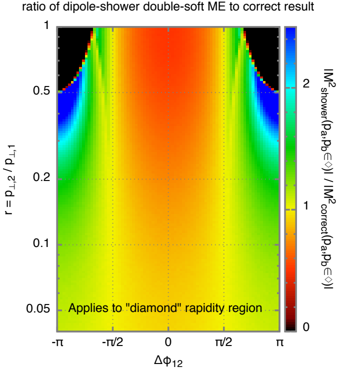
- Recently identified problem with standard dipole-like recoil

$$p_k^\mu = \left(1 - \frac{p_{ij}^2}{2\tilde{p}_{ij}\tilde{p}_k}\right)\tilde{p}_k^\mu$$

$$p_i^\mu = \tilde{z}\,\tilde{p}_{ij}^\mu + (1-\tilde{z})\frac{p_{ij}^2}{2\tilde{p}_{ij}\tilde{p}_k}\tilde{p}_k^\mu + k_\perp^\mu$$

$$p_j^\mu = (1-\tilde{z})\,\tilde{p}_{ij}^\mu + \tilde{z}\frac{p_{ij}^2}{2\tilde{p}_{ij}\tilde{p}_k}\tilde{p}_k^\mu - k_\perp^\mu$$

- Angular correlations across multiple emissions due to recoil on splitter in anti-collinear region
- Spoils $\alpha_s \to 0$ consistency check



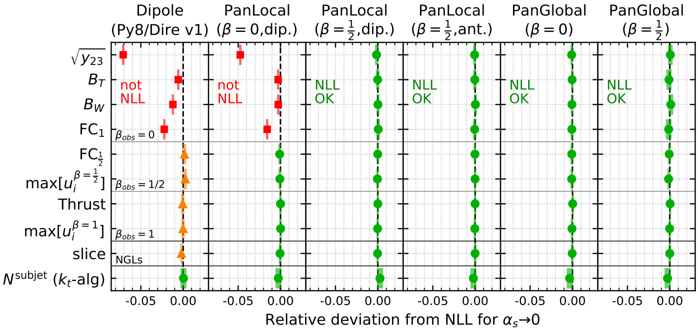
Momentum mapping in dipole-like showers

[Dasgupta,Dreyer,Hamilton,Monni,Salam,Soyez] arXiv:2002.11114

- Problem can be solved by partitioning of antenna radiation pattern and choosing a suitable evolution variable ($\beta \sim 1/2$)

$$k_T = \rho v e^{\beta|\bar{\eta}|} \qquad \rho = \left(\frac{s_i s_j}{Q^2 s_{ij}}\right)^{\beta/2}$$

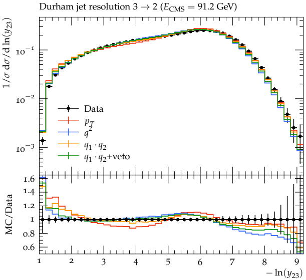
- Three different recoil schemes lead to NLL result if β chosen appropriately: Local dipole, local antenna, and global antenna
- NLL correct for global and non-global observables in $e^+e^- \to$hadrons

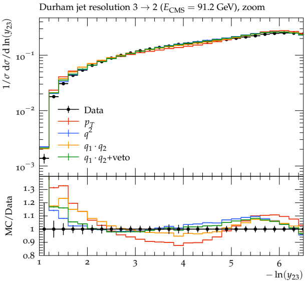


Momentum mapping in angular ordered showers

[Bewick,Ferrario-Ravasio,Richardson,Seymour] arXiv:1904.11866

- Recoil schemes affect logarithmic accuracy but impact also phase-space coverage
- In context of angular ordered Herwig 7 (NLL accurate for global observables)
  - q_T preserving scheme:
    Maintains logarithmic accuracy
    Overpopulates hard region
  - q^2 preserving scheme:
    Breaks logarithmic accuracy
    Good description of hard region
  - Dot product preserving scheme (new):
    Maintains logarithmic accuracy
    Good description of hard radiation





Analytic properties of branching equations

Forward vs. backward evolution

Properties of splitting kernels

- At any order of perturbation theory, splitting functions obey sum rules

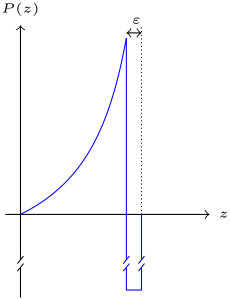
$$\int_0^1 \mathrm{d}\zeta\, \hat{P}_{qq}(\zeta) = 0 \qquad \rightarrow \qquad \text{flavor sum rule}$$

$$\sum_{c=q,g} \int_0^1 \mathrm{d}\zeta\, \zeta\, \hat{P}_{ac}(\zeta) = 0 \qquad \rightarrow \qquad \text{momentum sum rule}$$

$\rightarrow$ defines regularized splitting functions $\hat{P}_{ab}$ as

$$\hat{P}_{ab}(z) = \lim_{\varepsilon\to 0}\left[P_{ab}(z)\Theta(1-\varepsilon-z) - \delta_{ab}\frac{\Theta(z-1+\varepsilon)}{\varepsilon}\sum_{c=q,g}\int_0^{1-\varepsilon} d\zeta\, \zeta\, P_{ac}(\zeta)\right]$$

- What does that mean in physics terms?
  - Contribution $\propto \Theta(1-\varepsilon-z)$ corresponds to real-emission corrention
  - Contribution $\propto \Theta(z-1+\varepsilon)$ corresponds to virtual correction
  - Momentum sum rule is a unitarity constraint



Relation between parton shower and DGLAP evolution

- DGLAP equation for fragmentation functions

$$\frac{\mathrm{d}\, xD_a(x,t)}{\mathrm{d}\ln t} = \sum_{b=q,g} \int_0^1 \mathrm{d}\tau \int_0^1 \mathrm{d}z\, \frac{\alpha_s}{2\pi} \big[zP_{ab}(z)\big]_+ \tau D_b(\tau,t)\, \delta(x-\tau z)$$

- Refine plus prescription $\big[zP_{ab}(z)\big]_+ = \lim_{\varepsilon\to 0} zP_{ab}(z,\varepsilon)$

$$P_{ab}(z,\varepsilon) = P_{ab}(z)\,\Theta(1-\varepsilon-z) - \delta_{ab} \sum_{c\in\{q,g\}} \frac{\Theta(z-1+\varepsilon)}{\varepsilon} \int_0^{1-\varepsilon} \mathrm{d}\zeta\,\zeta\, P_{ac}(\zeta)$$

- Rewrite for finite ε

$$\frac{\mathrm{d}\ln D_a(x,t)}{\mathrm{d}\ln t} = -\sum_{c=q,g} \int_0^{1-\varepsilon} \mathrm{d}\zeta\,\zeta\, \frac{\alpha_s}{2\pi} P_{ac}(\zeta) + \sum_{b=q,g} \int_x^{1-\varepsilon} \frac{\mathrm{d}z}{z}\, \frac{\alpha_s}{2\pi}\, P_{ab}(z)\, \frac{D_b(x/z,t)}{D_a(x,t)}$$

- First term is derivative of Sudakov factor $\Delta = \exp\{-\lambda\}$

$$\Delta_a(t,Q^2) = \exp\left\{ -\int_t^{Q^2} \frac{\mathrm{d}\bar{t}}{\bar{t}} \sum_{c=q,g} \int_0^{1-\varepsilon} \mathrm{d}\zeta\,\zeta\, \frac{\alpha_s}{2\pi} P_{ac}(\zeta) \right\}$$

Relation between parton shower and DGLAP evolution

- Use generating function $\Pi_a(x,t,Q^2) = D_a(x,t)\Delta_a(t,Q^2)$ to write

$$\frac{\mathrm{d}\ln\Pi_a(x,t,Q^2)}{\mathrm{d}\ln t/Q^2} = \sum_{b=q,g}\int_x^{1-\varepsilon}\frac{\mathrm{d}z}{z}\,\frac{\alpha_s}{2\pi}\,P_{ab}(z)\,\frac{D_b(x/z,t)}{D_a(x,t)}\;.$$

- If hadron not resolved, obtain

$$\frac{\mathrm{d}}{\mathrm{d}\ln t/Q^2}\ln\left(\frac{\Pi_a(x,t,Q^2)}{D_a(x,t)}\right) = \frac{\mathrm{d}\Delta_a(t,Q^2)}{\mathrm{d}\ln t/Q^2} = \sum_{b=q,g}\int_0^{1-\varepsilon}\mathrm{d}z\,z\,\frac{\alpha_s}{2\pi}\,P_{ab}(z)$$

- Survival probabilities for one parton between scales t_1 and t_2:
  - $\dfrac{\Pi_a(x,t_2,Q^2)}{\Pi_a(x,t_1,Q^2)}$ Resolved hadron $\leftrightarrow$ constrained (backward) evolution
  - $\dfrac{\Delta_a(t_2,Q^2)}{\Delta_a(t_1,Q^2)}$ No resolved hadron $\leftrightarrow$ unconstrained (forward) evolution
- Parton-showers draw t_2-points starting from t_1 based on these probabilities

See heuristic introduction and tutorial for how to do this in practice

Matching and merging

Where parton showers meet higher orders

Toy model for IR subtraction at NLO

[Frixione,Webber] hep-ph/0204244

- Assume system of charges radiating "photons" of fractional energy x.
- Predicting observables at NLO amounts to computing expectation value

$$\langle O \rangle = \lim_{\varepsilon \to 0} \int_0^1 \mathrm{d}x \, x^{-2\varepsilon} \left[\left(\frac{\mathrm{d}\sigma}{\mathrm{d}x} \right)_{\mathrm{B}} O_0 + \left(\frac{\mathrm{d}\sigma}{\mathrm{d}x} \right)_{\mathrm{V}} O_0 + \left(\frac{\mathrm{d}\sigma}{\mathrm{d}x} \right)_{\mathrm{R}} O_1(x) \right]$$

- Born, virtual and real-emission contributions given by

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}x} \right)_{\mathrm{B,V,R}} = \quad \mathrm{B}\,\delta(x), \qquad \left(\mathrm{V}_f + \frac{\mathrm{B}\mathrm{V}_s}{2\varepsilon} \right) \delta(x), \qquad \frac{\mathrm{R}(x)}{x}$$

KLN cancellation theorem: $\lim_{x\to 0} \mathrm{R}(x) = \mathrm{B}\mathrm{V}_s$
Infrared safe observable: $\lim_{x\to 0} O_1(x) = O_0$

Virtual correction $\begin{cases} \mathrm{V}_f & - \quad \text{finite piece} \\ \mathrm{B}\mathrm{V}_s/2\varepsilon & - \quad \text{singular piece} \end{cases}$

Implicit: All higher-order terms proportional to coupling α

Toy model for IR subtraction at NLO

- Add and subtract approximation of real correction in soft limit

$$\langle O\rangle_R = \mathrm{BV}_s\, O(0) \int_0^1 \mathrm{d}x \frac{x^{-2\varepsilon}}{x} + \int_0^1 \mathrm{d}x\, \frac{\mathrm{R}(x)\, O(x) - \mathrm{BV}_s\, O(0)}{x^{1+2\varepsilon}}$$

- Second integral non-singular $\rightarrow$ set $\varepsilon = 0$

$$\langle O\rangle_R = -\frac{\mathrm{BV}_s}{2\varepsilon}\, O(0) + \int_0^1 \mathrm{d}x\, \frac{\mathrm{R}(x)\, O(x) - \mathrm{BV}_s\, O(0)}{x}$$

- Combine everything with Born and virtual correction

$$\langle O\rangle = \Big(\mathrm{B} + \mathrm{V}_f\Big)\, O(0) + \int_0^1 \frac{\mathrm{d}x}{x} \Big[\mathrm{R}(x)\, O(x) - \mathrm{BV}_s\, O(0)\Big]$$

  Both terms separately finite
- Rewrite for future reference

$$\langle O\rangle = \Big(\mathrm{B} + \mathrm{V} + \mathrm{I}\Big)\, O(0) + \int_0^1 \frac{\mathrm{d}x}{x} \Big[\mathrm{R}(x)\, O(x) - \mathrm{S}\, O(0)\Big]$$

  $\mathrm{I} = -\mathrm{BV}_s/2\varepsilon \rightarrow$ Integrated subtraction term
  $\mathrm{S} = \mathrm{BV}_s \rightarrow$ Real subtraction term

IR subtraction at NLO

- QCD subtraction more cumbersome:
  - Soft limit color dependent [Bassetto,Ciafaloni,Marchesini] PR100(1983)201

$$|\mathcal{M}(1,\ldots,j,\ldots,n)|^2 \xrightarrow{j\to\text{soft}} -\sum_{i,k\neq i} \frac{8\pi\mu^{2\varepsilon}\alpha_s}{p_i p_j}$$

$$\times\, {}_m\langle 1,\ldots,i,\ldots,k,\ldots,n| \frac{\mathbf{T}_i\mathbf{T}_k\; p_i p_k}{(p_i+p_k)p_j} |1,\ldots,i,\ldots,k,\ldots,n\rangle_m$$

  $\mathbf{T}_i$ - color insertion operator for parton i

  $|1,\ldots,i,\ldots,k,\ldots,n\rangle_m$ - m-parton Born amplitude

  - Collinear limit spin dependent [Altarelli,Parisi] NPB126(1977)298

$$|\mathcal{M}(1,\ldots,i,\ldots,j,\ldots,n)|^2 \xrightarrow{i,j\to\text{coll}} \frac{8\pi\mu^{2\varepsilon}\alpha_s}{2p_i p_j}$$

$$\times\, {}_m\langle 1,\ldots,ij,\ldots,n|\hat{P}_{(ij)i}(z,k_T,\varepsilon)\,|1,\ldots,ij,\ldots,n\rangle_m$$

  $\hat{P}_{(ij)i}(z,k_T,\varepsilon)$ - Spin-dependent DGLAP kernel

- Basic features surviving from toy model are phase-space mapping and subtraction terms as products of Born times splitting operator
- Commonly used techniques: Dipole method & FKS method

  [Catani,Seymour] NPB485(1997)291, [Catani,Dittmaier,Seymour,Trocsanyi] NPB627(2002)189
  [Frixione,Kunszt,Signer] NPB467(1996)399

Matching methods

Matching schemes

Two major techniques to match NLO calculations and parton showers

Additive (MC@NLO-like)

[Frixione,Webber] hep-ph/0204244

- Use parton-shower splitting kernel as an NLO subtraction term
- Multiply LO event weight by Born-local K-factor including integrated subtraction term and virtual corrections
- Add hard remainder function consisting of subtracted real-emission correction

Multiplicative (POWHEG-like)

[Nason] hep-ph/0409146

- Use matrix-element corrections to replace parton-shower splitting kernel by full real-emission matrix element in first shower branching
- Multiply LO event weight by Born-local NLO K-factor (integrated over real corrections that can be mapped to Born according to PS kinematics)

Toy model for modified subtraction

[Frixione,Webber] hep-ph/0204244

- Revisit toy model for NLO

$$\langle O \rangle = \Big(\mathrm{B} + \mathrm{V} + \mathrm{I}\Big)\, O(0) + \int_0^1 \frac{\mathrm{d}x}{x} \Big[\mathrm{R}(x)\, O(x) - \mathrm{S}\, O(0)\Big]$$

- In parton showers, any number of “photons” can be emitted
- Emission probability controlled by Sudakov form factor

$$\Delta(x_1, x_2) = \exp\left\{ - \int_{x_1}^{x_2} \frac{\mathrm{d}x}{x}\, \mathrm{K}(x) \right\}$$

Evolution kernel behaves as $\lim_{x\to 0} \mathrm{K}(x) = \lim_{x \to 0} \mathrm{R}(x)/\mathrm{B} = \mathrm{V}_s$

- Define generating functional

$$\mathcal{F}^{(n)}_{\mathrm{MC}}(x, O) = \Delta(x_0, x)\, O_n(x) + \int_{x_0}^{x} \frac{\mathrm{d}\bar{x}}{\bar{x}} \frac{\mathrm{d}\Delta(\bar{x}, x)}{\mathrm{d}\ln \bar{x}}\, \mathcal{F}^{(n+1)}_{\mathrm{MC}}(\bar{x}, O)$$

- $\mathcal{F}^{(n)}_{\mathrm{MC}}(x, O)$ now replaces observable O $\rightarrow$ Naively:

  $O(0) \Leftrightarrow$ start MC with 0 emissions $\rightarrow \mathcal{F}^{(0)}_{\mathrm{MC}}(1, O)$

  $O(x) \Leftrightarrow$ start MC with 1 emission $\rightarrow \mathcal{F}^{(1)}_{\mathrm{MC}}(x, O)$

Toy model for modified subtraction

- Combined generating functional would be

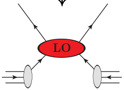
$$\left[\Big(\mathrm{B}+\mathrm{V}+\mathrm{I}\Big)-\int_0^1\frac{\mathrm{d}x}{x}\mathrm{S}\right]\mathcal{F}_{\mathrm{MC}}^{(0)}(1,O)+\int_0^1\frac{\mathrm{d}x}{x}\,\mathrm{R}(x)\,\mathcal{F}_{\mathrm{MC}}^{(1)}(x,O)$$

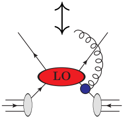
- This is wrong because

$$\mathcal{F}_{\mathrm{MC}}^{(0)}(O)=\Delta(x_c,1)\,O(0)+\int_{x_c}^1\frac{\mathrm{d}x}{x}\mathrm{K}(x)\Delta(x,1)\,O(x)+\ldots$$

- So $\mathrm{B}\,\mathcal{F}_{\mathrm{MC}}^{(0)}$ generates an $\mathcal{O}(\alpha)$ term that spoils NLO accuracy

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}x}\right)_{\mathrm{MC}}O(x)=\mathrm{B}\left[-\frac{\mathrm{K}(x)}{x}\,O(0)+\frac{\mathrm{K}(x)}{x}\,O(x)\right]$$





Toy MC@NLO

[Frixione,Webber] hep-ph/0204244

- The proper matching is obtained by subtracting this $\mathcal{O}(\alpha)$ contribution

$$\langle O \rangle = \underbrace{\left[\left(\mathrm{B} + \mathrm{V} + \mathrm{I} \right) + \int_0^1 \frac{\mathrm{d}x}{x} \left(\mathrm{BK}(x) - \mathrm{S} \right) \right]}_{\text{NLO-weighted Born cross section}} \mathcal{F}_{\mathrm{MC}}^{(0)}(1, O)$$

$$+ \int_0^1 \frac{\mathrm{d}x}{x} \underbrace{\left[\mathrm{R}(x) - \mathrm{BK}(x) \right]}_{\text{hard remainder}} \mathcal{F}_{\mathrm{MC}}^{(1)}(x, O)$$

- Like at fixed order, both terms are separately finite
- We call events from the first term $\mathbb{S}$-events ($\mathbb{S}$tandard) and events from the second term $\mathbb{H}$-events ($\mathbb{H}$ard)
- For further reference, define $\mathrm{D}^{(\mathrm{K})}(x) := \mathrm{BK}(x)$ as well as

$$\bar{\mathrm{B}}^{(\mathrm{K})} = \left(\mathrm{B} + \mathrm{V} + \mathrm{I} \right) + \int_0^1 \frac{\mathrm{d}x}{x} \left(\mathrm{D}^{(\mathrm{K})}(x) - \mathrm{S} \right) , \quad \mathrm{H}^{(\mathrm{K})}(x) = \mathrm{R}(x) - \mathrm{D}^{(\mathrm{K})}(x)$$

$\rightarrow$ compact notation

$$\langle O \rangle = \bar{\mathrm{B}}^{(\mathrm{K})} \mathcal{F}_{\mathrm{MC}}^{(0)}(O) + \int_0^1 \frac{\mathrm{d}x}{x} \mathrm{H}^{(\mathrm{K})}(x) \mathcal{F}_{\mathrm{MC}}^{(1)}(x, O)$$

Modified subtraction in QCD

[Frixione,Webber] hep-ph/0204244

- Leading-order calculation for observable O

$$\langle O\rangle = \int \mathrm{d}\Phi_B\, \mathrm{B}(\Phi_B)\, O(\Phi_B)$$

- NLO calculation for same observable

$$\langle O\rangle = \int \mathrm{d}\Phi_B \left\{ \mathrm{B}(\Phi_B) + \tilde{V}(\Phi_B) \right\} O(\Phi_B) + \int \mathrm{d}\Phi_R\, \mathrm{R}(\Phi_R)\, O(\Phi_R)$$

- Parton-shower result until first emission

$$\langle O\rangle = \int \mathrm{d}\Phi_B\, \mathrm{B}(\Phi_B) \left[\Delta^{(\mathrm{K})}(t_c)\, O(\Phi_B) + \int_{t_c} \mathrm{d}\Phi_1\, \mathrm{K}(\Phi_1)\, \Delta^{(\mathrm{K})}(t(\Phi_1))\, O(\Phi_R) \right]$$

$$\overset{\mathcal{O}(\alpha_s)}{\to} \int \mathrm{d}\Phi_B\, \mathrm{B}(\Phi_B) \left\{ 1 - \int_{t_c} \mathrm{d}\Phi_1 \mathrm{K}(\Phi_1) \right\} O(\Phi_B) + \int_{t_c} \mathrm{d}\Phi_B \mathrm{d}\Phi_1\, \mathrm{B}(\Phi_B)\, \mathrm{K}(\Phi_1)\, O(\Phi_R)$$

Phase space: $\mathrm{d}\Phi_1 = \mathrm{d}t\, \mathrm{d}z\, \mathrm{d}\phi$

Splitting functions: $\mathrm{K}(t,z) \to \alpha_s/(2\pi t) \sum \mathrm{P}(z)\, \Theta(\mu_Q^2 - t)$

Sudakov factors: $\Delta^{(\mathrm{K})}(t) = \exp\left\{ -\int_t \mathrm{d}\Phi_1 \mathrm{K}(\Phi_1) \right\}$

Modified subtraction in QCD

- Subtract $\mathcal{O}(\alpha_s)$ PS terms from NLO result

$$\int \mathrm{d}\Phi_B \left\{ \mathrm{B}(\Phi_B) + \tilde{\mathrm{V}}(\Phi_B) + \mathrm{B}(\Phi_B) \int \mathrm{d}\Phi_1 \mathrm{K}(\Phi_1) \right\} \ldots$$

$$+ \int \mathrm{d}\Phi_R \left\{ \mathrm{R}(\Phi_R) - \mathrm{B}(\Phi_B)\, \mathrm{K}(\Phi_1) \right\} \ldots$$

- In DLL approximation both terms finite $\rightarrow$
  MC events in two categories, $\mathbb{S}$tandard and $\mathbb{H}$ard

$$\mathbb{S} \;\rightarrow\; \bar{\mathrm{B}}^{(\mathrm{K})}(\Phi_B) = \mathrm{B}(\Phi_B) + \tilde{\mathrm{V}}(\Phi_B) + \mathrm{B}(\Phi_B) \int \mathrm{d}\Phi_1 \mathrm{K}(\Phi_1)$$

$$\mathbb{H} \;\rightarrow\; \mathrm{H}^{(\mathrm{K})} = \mathrm{R}(\Phi_R) - \mathrm{B}(\Phi_B)\mathrm{K}(\Phi_1)$$

- Color & spin correlations $\rightarrow$ NLO subtraction needed
  $1/N_c$ corrections can be faded out in soft region by smoothing function

$$\bar{\mathrm{B}}^{(\mathrm{K})}(\Phi_B) = \mathrm{B}(\Phi_B) + \tilde{\mathrm{V}}(\Phi_B) + \mathrm{I}(\Phi_B) + \int \mathrm{d}\Phi_1 \Big[\mathrm{B}(\Phi_B)\, \mathrm{K}(\Phi_1) - \mathrm{S}(\Phi_R) \Big]\, f(\Phi_1)$$

$$\mathrm{H}^{(\mathrm{K})}(\Phi_R) = \Big[\mathrm{R}(\Phi_R) - \mathrm{B}(\Phi_B)\, \mathrm{K}(\Phi_1) \Big]\, f(\Phi_1)$$

Dealing with color and spin

Method 1 [Frixione,Webber] hep-ph/0204244

- $f(\Phi_1) \to 0$ in soft-gluon limit
- Full NLO in hard / collinear region
- Subleading color terms not ϕ_1-dependent in soft domain

Method 2 [Krauss,Schönherr,Siegert,SH] arXiv:1111.1220

- Replace $\mathrm{B}(\Phi_B)\mathrm{K}(\Phi_1) \to \mathrm{S}(\Phi_R)$, includes color & spin correlations
- Can lead to non-probabilistic $\Delta^{(\mathrm{S})}(t)$
  $\to$ requires modification of veto algorithm

MC@NLO

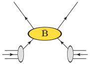
[Frixione,Webber] hep-ph/0204244

- Add parton shower, described by generating functional $\mathcal{F}_{\mathrm{MC}}$

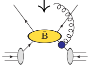
$$\langle O \rangle = \int \mathrm{d}\Phi_B \, \bar{\mathrm{B}}^{(\mathrm{K})}(\Phi_B) \, \mathcal{F}^{(0)}_{\mathrm{MC}}(\mu_Q^2, O) + \int \mathrm{d}\Phi_R \, \mathrm{H}^{(\mathrm{K})}(\Phi_R) \, \mathcal{F}^{(1)}_{\mathrm{MC}}(t(\Phi_R), O)$$

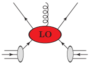
  Probability conservation: $\mathcal{F}_{\mathrm{MC}}(t, 1) = 1 \rightarrow$ cross section correct at NLO
- Expansion of matched result until first emission

$$\langle O \rangle = \int \mathrm{d}\Phi_B \bar{\mathrm{B}}^{(\mathrm{K})}(\Phi_B) \Big[\Delta^{(\mathrm{K})}(t_c) \, O(\Phi_B) \quad \leftrightarrow$$



$$+ \int_{t_c} \mathrm{d}\Phi_1 \mathrm{K}(\Phi_1) \Delta^{(\mathrm{K})}(t(\Phi_1)) \, O(\Phi_r) \Big] + \int \mathrm{d}\Phi_R \, \mathrm{H}^{(\mathrm{K})}(\Phi_{n+1}) \, O(\Phi_R)$$





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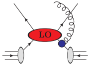
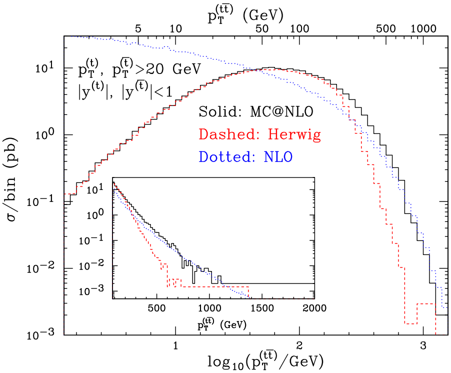


- Parametrically $\mathcal{O}(\alpha_s)$ correct
- Preserves logarithmic accuracy of PS

MC@NLO – Features

[Nason,Webber] arXiv:1202.1251

$pp \to t\bar{t} + X$ @ 14 TeV

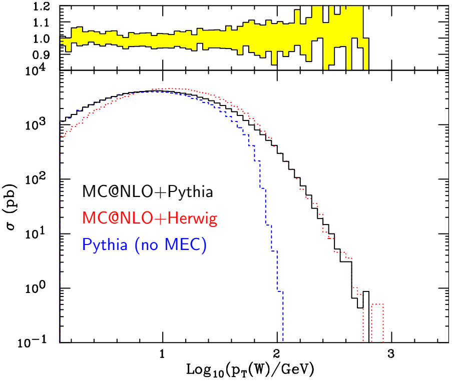


- MC@NLO interpolates smoothly between real-emission ME and PS

MC@NLO – Features

[Torrielli,Frixione] arXiv:1002.4293

$pp \to W + X$ @ 14 TeV



- MC@NLO with different PS agree at high $p_T \leftrightarrow$ NLO
- Differences at low p_T due to differences in PS

POWHEG

[Nason] hep-ph/0409146

- Aim of the method: Eliminate negative weights from MC@NLO
- Replace $\mathrm{BK} \to \mathrm{R} \Rightarrow$ no $\mathbb{H}$-events $\quad \Rightarrow \quad \bar{\mathrm{B}}^{(\mathrm{R})}$ positive in physical region
- Expectation value of observable is

$$\langle O \rangle = \int \mathrm{d}\Phi_B \bar{\mathrm{B}}^{(\mathrm{R})}(\Phi_B) \Bigg[\Delta^{(\mathrm{R})}(t_c, s_{\mathrm{had}})\, O(\Phi_B) + \int_{t_c}^{s_{\mathrm{had}}} \mathrm{d}\Phi_1 \frac{\mathrm{R}(\Phi_R)}{\mathrm{B}(\Phi_B)} \Delta^{(\mathrm{R})}(t(\Phi_1), s_{\mathrm{had}})\, O(\Phi_R) \Bigg]$$

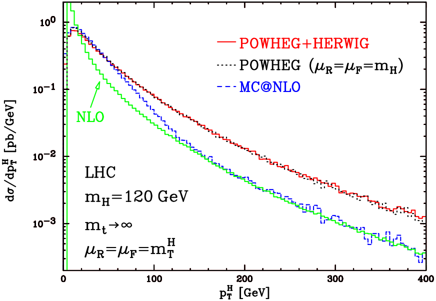
- μ_Q^2 has changed to hadronic centre-of-mass energy squared, s_{had}, as full phase space for real-emission correction, R, must be covered
- Absence of $\mathbb{H}$-events leads to enhancement of high-p_T region by

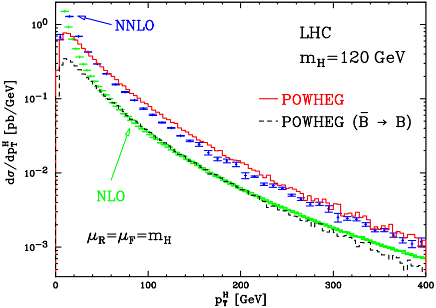
  $$K = \frac{\bar{\mathrm{B}}}{\mathrm{B}} = 1 + \mathcal{O}(\alpha_s)$$

  Formally beyond NLO, but sizeable corrections in practice

POWHEG – Features

[Alioli,Nason,Oleari,Re] arXiv:0812.0578





- Large enhancement at high $p_{T,h}$
- Can be traced back to large NLO correction
- Fortunately, NNLO correction is also large $\to$ $\sim$ agreement

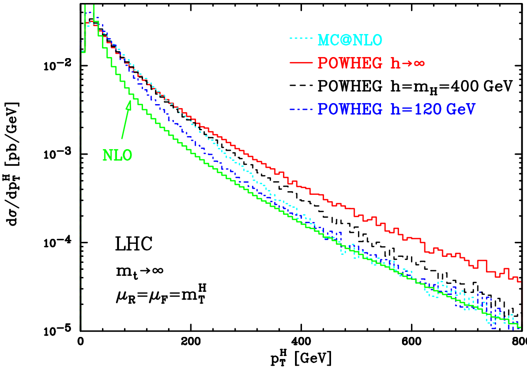
Improved POWHEG

- To avoid problems in high-p_T region, split real-emission ME into singular and finite parts as $\mathrm{R} = \mathrm{R}^s + \mathrm{R}^f$
- Treat singular piece in $\mathbb{S}$-events and finite piece in $\mathbb{H}$-events
  Similar to MC@NLO with redefined PS evolution kernels
- Differential event rate up to first emission

$$\langle O \rangle = \int \mathrm{d}\Phi_B \bar{\mathrm{B}}^{(\mathrm{R}^\mathrm{s})}(\Phi_B) \bigg[\Delta^{(\mathrm{R}^\mathrm{s})}(t_c, s_\mathrm{had})\, O(\Phi_B) + \int_{t_c}^{s_\mathrm{had}} \mathrm{d}\Phi_1 \frac{\mathrm{R}^s(\Phi_R)}{\mathrm{B}(\Phi_B)} \Delta^{(\mathrm{R}^\mathrm{s})}(t(\Phi_1), s_\mathrm{had})\, O(\Phi_R) \bigg] + \int \mathrm{d}\Phi_R\, \mathrm{R}^f_n(\Phi_R)$$

POWHEG – Features

[Alioli,Nason,Oleari,Re] arXiv:0812.0578



- Singular real-emission part here defined as

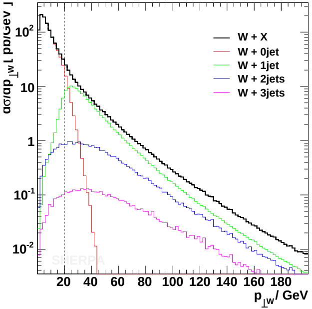
$$\mathrm{R}^s = \mathrm{R}\,\frac{h^2}{p_T^2 + h^2}$$

- Can "tune" NNLO contribution by varying free parameter h

Multi-jet merging

Basic idea of merging

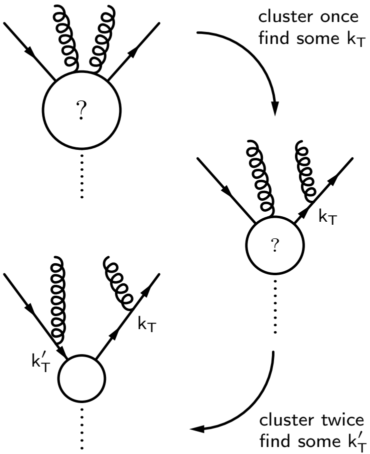
- Separate phase space into "hard" and "soft" region
- Parton shower populates soft domain
- N^{x}LO real corrections replace PS emission term in hard domain
- Need criterion to define "hard" & "soft" $\rightarrow$ jet measure Q and corresponding cut, $Q_{\rm cut}$



Parton shower histories

[André,Sjöstrand] hep-ph/9708390

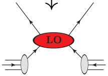
- Start with some "core" process for example $e^+e^- \to q\bar{q}$
- This process is considered inclusive It sets the resummation scale μ_Q^2
- Higher-multiplicity ME can be reduced to core by clustering
  - Identify most likely splitting according to PS emission probability
  - Combine partons into mother according to PS kinematics
  - Continue until core process reached



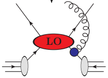
Basic idea of merging

- MC@LO split into $Q < Q_{\mathrm{cut}}$ (PS) and $Q > Q_{\mathrm{cut}}$ (ME) region
  PS expression replaced by real-emission matrix-element in ME region

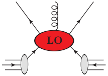
$$\langle O \rangle = \int \mathrm{d}\Phi_B \mathrm{B}(\Phi_B) \Big[\Delta^{(\mathrm{K})}(t_c, \mu_Q^2)\, O(\Phi_B)$$



$$+ \int_{t_c}^{\mu_Q^2} \mathrm{d}\Phi_1 \mathrm{K}(\Phi_1)\, \Delta^{(\mathrm{K})}(t(\Phi_1), \mu_Q^2)\, \Theta(Q_{\mathrm{cut}} - Q)\, O(\Phi_R) \Big]$$



$$+ \int \mathrm{d}\Phi_R \mathrm{R}(\Phi_R)\, \Delta^{(\mathrm{K})}(t(\Phi_R), \mu_Q^2)\, \Theta(Q - Q_{\mathrm{cut}})\, O(\Phi_R)$$



- Jet veto in PS / Jet cut on ME
- To match $\mathrm{K}(\phi_1)$, weight $\mathrm{R}(\phi_1)$ by $\alpha_s(k_T^2)/\alpha_s(\mu_R^2)$

Truncated vetoed parton showers

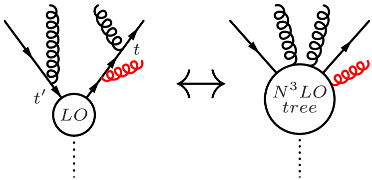
[Lönnblad] hep-ph/0112284

- In hard region $\Delta(t(\Phi_R), \mu_Q^2)$ is additional weight
- Most efficiently computed using pseudo-showers

  Recall PS no-emission probability: Constrained: $\Pi(x, t_2, \mu_Q^2)/\Pi(x, t_1, \mu_Q^2)$

  Unconstrained: $\Delta(t_2, \mu_Q^2)/\Delta(t_1, \mu_Q^2)$

- Start PS from core process
- Evolve until predefined branching $\leftrightarrow$ truncated parton shower
- Emissions that would produce additional hard jets lead to event rejection (veto)



Truncated unvetoed parton showers

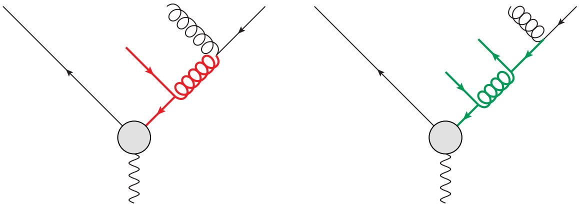
[Nason] hep-ph/0409146

- For $t \neq Q$, PS may generate emissions between μ_Q^2 and $t(\Phi_R)$, as

$$\Delta(t,\mu_Q^2) = \Delta(t,\mu_Q^2;>Q_{\rm cut})\,\Delta(t,\mu_Q^2;<Q_{\rm cut})$$

$$\Delta(t,\mu_Q^2;>Q_{\rm cut}) = \exp\left\{-\int_t^{\mu_Q^2} {\rm d}\Phi_1\,K(\Phi_1)\,\Theta(Q-Q_{cut})\right\}$$

- Momentum and flavor conserving implementation non-trivial
  Example: Two emissions may be allowed, while one may be not

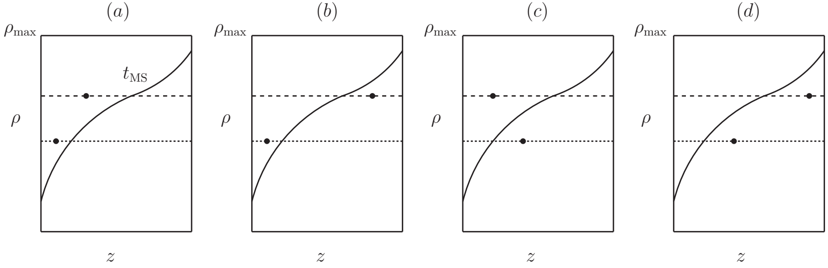


- Effects of non-trivial terms formally suppressed
  Better algorithm may be easier to implement

Evading truncated unvetoed parton showers

[Lönnblad] hep-ph/0112284

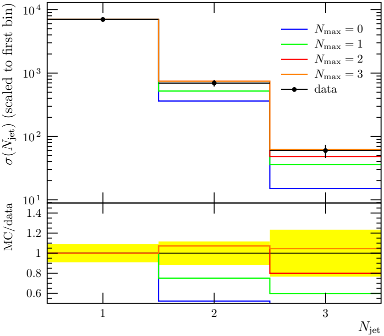
- Generate truncated unvetoed configurations with parton shower
  effective redefinition of Q, assuming PS ordering parameter $\sim$ "hardness"
- Schematic illustration of phase space coverage

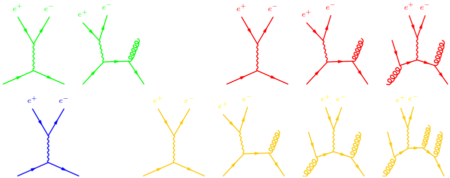


from 2-jet ME | from 0-jet ME | from 1-jet ME | from 0-jet ME

- Straightforward implementation, no reshuffling of kinematics or flavor

Effects of merging - Z+jets at the Tevatron





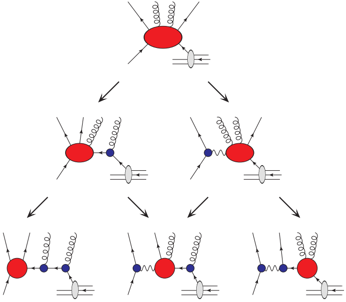
- MC predictions for exclusive n-jet rates match data well as long as corresponding final states are described by matrix elements

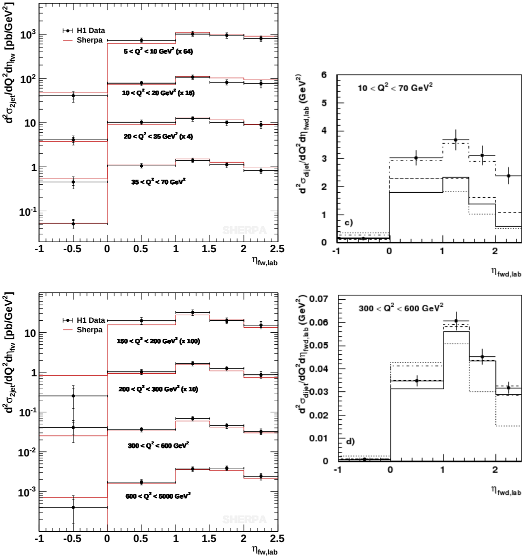
Lessons from HERA

[Carli,Gehrmann,SH] arXiv:0912.3715

Simulation often too focused on resonant contributions

Need be inclusive to describe DIS, low-mass Drell-Yan or photon / diphoton production





Unitarization

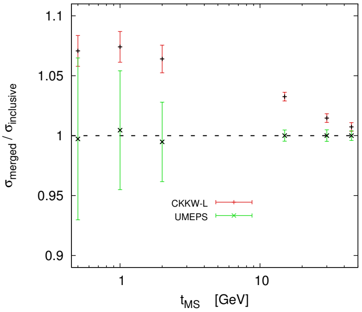
[Lönnblad,Prestel] arXiv:1211.4827, [Plätzer] arXiv:1211.5467
[Bellm,Gieseke,Plätzer] arXiv:1705.06700

- Unitarity condition of PS:

$$1 = \Delta^{(\mathrm{K})}(t_c) + \int_{t_c} \mathrm{d}\Phi_1\, \mathrm{K}(\Phi_1)\, \Delta^{(\mathrm{K})}(t)$$

- ME+PS(@NLO) violates PS unitarity as ME ratio replaces splitting kernels in emission terms, but not in Sudakovs

$$\mathrm{K}(\Phi_1) \to \frac{\mathrm{R}(\Phi_1, \Phi_B)}{\mathrm{B}(\Phi_B)}$$



- Can be corrected by explicit subtraction

$$1 = \underbrace{\left\{\Delta^{(\mathrm{K})}(t_c) + \int_{t_c} \mathrm{d}\Phi_1 \left[\mathrm{K}(\Phi_1) - \frac{\mathrm{R}(\Phi_1, \Phi_B)}{\mathrm{B}(\Phi_B)}\right] \Theta(Q - Q_{\mathrm{cut}})\, \Delta^{(\mathrm{K})}(t)\right\}}_{\text{unresolved emission / virtual correction}}$$

$$+ \underbrace{\int_{t_c} \mathrm{d}\Phi_1 \left[\mathrm{K}(\Phi_1)\Theta(Q_{\mathrm{cut}} - Q) + \frac{\mathrm{R}(\Phi_1, \Phi_B)}{\mathrm{B}(\Phi_B)}\Theta(Q - Q_{\mathrm{cut}})\right] \Delta^{(\mathrm{K})}(t)}_{\text{resolved emission}}$$

Merging of multiple matched calculations

- ME+PS merging for 0+1-jet in MC@NLO notation

$$\langle O\rangle = \int \mathrm{d}\Phi_B \mathrm{B}(\Phi_B)\Bigg[\Delta^{(\mathrm{K})}(t_c)\,O(\Phi_B) + \int_{t_c} \mathrm{d}\Phi_1 \mathrm{K}(\Phi_1)\,\Delta^{(\mathrm{K})}(t)\,\Theta(Q_{\mathrm{cut}} - Q)\,O(\Phi_R)\Bigg]$$
$$+ \int \mathrm{d}\Phi_R\,\mathrm{R}(\Phi_R)\,\Delta^{(\mathrm{K})}(t(\Phi_R),\mu_Q^2)\,\Theta(Q - Q_{\mathrm{cut}})\,O(\Phi_R)$$

- Reorder by parton multiplicity k, change notation $\mathrm{R}_k \to \mathrm{B}_{k+1}$
- Analyze exclusive contribution from k hard partons only $(t_0 = \mu_Q^2)$

$$\langle O\rangle_k^{\mathrm{excl}} = \int \mathrm{d}\Phi_k\,\mathrm{B}_k \prod_{i=0}^{k-1} \Delta_i^{(\mathrm{K})}(t_{i+1},t_i)\,\Theta(Q_k - Q_{\mathrm{cut}})$$
$$\times \Bigg[\Delta_k^{(\mathrm{K})}(t_c,t_k)\,O_k + \int_{t_c}^{t_k} \mathrm{d}\Phi_1\,\mathrm{K}_k\,\Delta_k^{(\mathrm{K})}(t_{k+1},t_k)\,\Theta(Q_{\mathrm{cut}} - Q_{k+1})\,O_{k+1}\Bigg]$$

Merging of multiple matched calculations

[Lavesson,Lönnblad,Prestel] arXiv:0811.2912 arXiv:1211.7278
[Gehrmann,Krauss,Schönherr,Siegert,SH] arXiv:1207.5031 arXiv:1207.5030
[Frederix,Frixione] arXiv:1209.6215

- Analyze exclusive contribution from k hard partons

$$\begin{aligned}\langle O\rangle_k^{\text{excl}} \;=\; &\int \mathrm{d}\Phi_k\, \bar{\mathrm{B}}_k^{(\mathrm{K})} \prod_{i=0}^{k-1} \Delta_i^{(\mathrm{K})}(t_{i+1},t_i)\,\Theta(Q_k-Q_{\text{cut}})\\
&\times\left(1+\frac{\mathrm{B}}{\bar{\mathrm{B}}_k^{(\mathrm{K})}}\sum_{i=0}^{k-1}\int_{t_{i+1}}^{t_i}\mathrm{d}\Phi_1\mathrm{K}_i\,\Theta(Q_i-Q_{\text{cut}})+\ldots\right)\\
&\times\left[\Delta_k^{(\mathrm{K})}(t_c,t_k)\,O_k\;+\;\int_{t_c}^{t_k}\mathrm{d}\Phi_1\;\mathrm{K}_k\,\Delta_k^{(\mathrm{K})}(t_{k+1},t_k)\,\Theta(Q_{\text{cut}}-Q_{k+1})\,O_{k+1}\right]\\
&+\int \mathrm{d}\Phi_{k+1}\,\mathrm{H}_k^{(\mathrm{K})}\,\Delta_k^{(\mathrm{K})}(t_k,\mu_Q^2)\,\Theta(Q_k-Q_{\text{cut}})\,\Theta(Q_{\text{cut}}-Q_{k+1})\,O_{k+1}\end{aligned}$$

- Born matrix element $\rightarrow$ NLO-weighted Born
- Add hard remainder function
- Subtract $\mathcal{O}(\alpha_s)$ terms from truncated vetoed PS

A different perspective on NLO merging

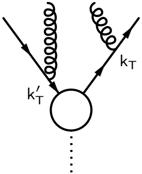
- Define compound evolution kernel

$$\tilde{\mathrm{K}}_k(\Phi_{k+1}) = \mathrm{K}_k(\Phi_{k+1})\,\Theta(t_k - t_{k+1}) + \sum_{i=n}^{k-1} \mathrm{K}_i(\Phi_i)\,\Theta(t_i - t_{k+1})\Theta(t_{k+1} - t_{i+1})$$

- Extend modified subtraction

$$\tilde{\mathrm{B}}_k^{(\mathrm{K})}(\Phi_k) = \Big[\mathrm{B}_k(\Phi_k) + \tilde{\mathrm{V}}_k(\Phi_k) + \mathrm{I}_k(\Phi_k)\Big] + \int \mathrm{d}\Phi_1 \Big[\mathrm{B}_k(\Phi_k)\tilde{\mathrm{K}}_k(\Phi_1) - \mathrm{S}_k(\Phi_{k+1})\Big]$$

$$\tilde{\mathrm{H}}_k^{(\mathrm{K})}(\Phi_{k+1}) = \mathrm{R}_k(\Phi_{k+1}) - \mathrm{B}_k(\Phi_k)\tilde{\mathrm{K}}_k(\Phi_1)$$



- Differential event rate for exclusive $n+k$-jet events

$$\langle O\rangle_k^{\mathrm{excl}} = \int \mathrm{d}\Phi_k\,\tilde{\mathrm{B}}_k^{(\mathrm{D})}\,\Theta(Q_k - Q_{\mathrm{cut}}) \times \left[\tilde{\Delta}_k^{(\mathrm{K})}(t_c,\mu_Q^2)\,O_k + \int_{t_c}^{\mu_Q^2} \mathrm{d}\Phi_1\,\tilde{\mathrm{K}}_k\,\tilde{\Delta}_k^{(\mathrm{K})}(t,\mu_Q^2)\,\Theta(Q_{\mathrm{cut}} - Q_{k+1})\,O_{k+1}\right] + \int \mathrm{d}\Phi_{k+1}\,\tilde{\mathrm{H}}_k^{(\mathrm{D})}\,\tilde{\Delta}_k^{(\mathrm{K})}(t_{k+1},\mu_Q^2)\,\Theta(Q_{\mathrm{cut}} - Q_{k+1})$$

Summary of this lecture

- Event generators are a topic of intense research, and they are expected to remain so as they provide the only effective means to simulate fully differential events with QCD radiation at both high and low scales
- The comparison with analytic resummation provides provides new, important constraints on old algorithms. Away from the $\alpha_s \to 0$ limit differences appear due to momentum and probability conservation
- The extension of parton showers to higher perturbative orders and to higher logarithmic accuracy as well as higher accuracy in the $1/N_c$ expansion will be an important step towards high-precision event simulation at the HL-LHC and future colliders
- A close interplay with AI/ML will help to accelerate old algorithms and develop new methods to provide cutting edge tools for phenomenology